SEMESTER-V

MAJOR PAPER-VIII

REAL ANALYSIS-I

SUBJECT CODE: U5MA01MT

UNIT-1: FUNCTIONS AND SEQUENCES

Functions- real valued functions- equivalence- countability and real numbers- least upper bound- definition of sequence and subsequence- limit of a sequence- convergent sequence

ch-1.4 to 1.7, 2.1 to 2.3 of Goldberg.

UNIT-II: SEQUENCES (contd)..

Divergent sequences- Bounded sequences- Monotone sequence- Operations on convergent sequences- operation on divergent sequences- Limit superior and Limit inferior- cauchy sequences ch- 2.4 to 2.10 of Goldberg.

UNIT-III: SERIES OF REAL NUMBERS

Convergence and Divergence- Series with non negative terms- Alternating series- conditional convergence and Absolute convergence- Test for Absolute convergence.

ch- 3.1 to 3.4 and 3.6 of Goldberg.

UNIT-IV: SERIES OF REAL NUMBERS (contd.)

Test for Absolute convergence - the class ℓ^2 - Limit of a function on the real line - Metric Spaces - Limits in metric spaces.

ch - 3.7, 3.10, 4.1 to 4.3 of Goldberg.

UNIT-V: CONTINUOUS FUNCTIONS ON METRIC SPACES

Functions continuous at a point on the real line - Reformulation - Functions continuous on a metric spaces - open sets - closed sets.

ch. 5.1 to 5.5 of Goldberg.

Recommended Text:

R. Goldberg, Methods of Real Analysis Oxford & IBH Publishing co., New Delhi (II Edition) 2012.

Reference Books:

Tom M. Apostol, Mathematical Analysis, Addison-Wesley New York (II Edition) 2002.

Sanjay Arora and Bansi Lal, Introduction to Real Analysis, Satya Prakashan, New Delhi (II Edition) 2008.

Malik S.C and Savita Arora, Mathematical Analysis, Wiley Eastern Limited, New Delhi (II Edition) 2010

Bartle R.G and Shebert, Real Analysis, John Wiley and Sons Inc., New York (III Edition) 2012

UNIT - 1

FUNCTIONS AND SEQUENCES

Introduction:

1. UNION OF SETS:

If A and B are sets, then $A \cup B$ is the set of all elements in either A or B (or both).

(i.e.) $A \cup B = \{x / x \in A \text{ or } x \in B\}$

Ex: If $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$

then $A \cup B = \{1, 2, 3, 4, 5\}$

2. INTERSECTION OF SETS:

If A and B are sets, then $A \cap B$ is the set of all elements in both A and B.

(i) $A \cap B = \{x / x \in A \text{ and } x \in B\}$

Ex: If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ then,

$A \cap B = \{2, 3\}$

3. EMPTY SETS:

The empty set (denoted by ϕ) as the set which has no elements.

Thus $\{1, 2\} \cap \{3, 4\} = \phi$

Note: For any set A.

$$A \cup \phi = A \text{ and } A \cap \phi = \phi$$

4. B Minus A :

If A and B are sets, then $B-A$ is the set of all elements of B which are not elements of A.

(i.e) $B-A = \{x \mid x \in B, x \notin A\}$

5.

If every element of the set A is an element of the set B, we write $A \subset B$ or $B \supset A$. If $A \subset B$, we say that A is a subset of B.

A proper subset of B is a subset

$A \subset B \ni : A \neq B$

thus, $A = \{1, 6, 7\}$, $B = \{1, 3, 6, 7, 8\}$,

$C = \{2, 3, 4, 5, \ldots\ldots, 100\}$, then $A \subset B$ but $B \not\subset C$ (even though C has 99 elements and B has only 5)

6. EQUAL SETS :

Two sets are said to be equal if they contain the same elements.

thus $A = B$ iff $A \subset B$ and $B \subset A$.

Theorem: 1

If A, B are subsets of S. then

$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

Proof:

(i) $(A \cup B)' = A' \cap B'$

If $x \in (A \cup B)'$

then $x \notin A \cup B$

Thus x is an element of neither A nor B.

So that, $x \in A'$ and $x \in B'$

$x \in A' \cap B'$

Hence $(A \cup B)' \subset A' \cap B'$ ----------→ (1)

Conversely, if $y \in A' \cap B'$

then $y \in A'$ and $y \in B'$

So that, $y \notin A$ and $y \notin B$

Thus, $y \notin A \cup B$

$y \in (A \cup B)'$

Hence $A' \cap B' \subset (A \cup B)'$ ----------→ (2)

From (1) & (2)

$(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Similarly proved as in (i)

Cartesian Product of two sets:

The cartesian product of two non-empty sets A and B, denoted by $A \times B$ is the

Set of all ordered pairs $(a,b) \ni : a \in A$ and $b \in B$.

(i.e) $A \times B = \{(a,b) : a \in A, b \in B\}$

Functions: 1.4

Definition: Let A and B be any two sets. A function f from (or on) A into B is a subset of $A \times B$ with the property that each $a \in A$ belongs to one pair (a,b).

Instead of $(x,y) \in f$, we usually write $y = f(x)$. then y is called the image of x under f. the set A is called the domain of f. the range of f is the set $\{b \in B \mid b = f(a)$ for some $a\}$.

(i.e) The range of f is the subset of B consisting of all images of elements of A. Such a function is sometimes called a mapping of A into B.

If $C \subset B$, then $f^{-1}(C)$ is defined as $\{a \in A \mid f(a) \in C\}$, the set of all points in the domain of f whose images are in C. If C has only one point in it, say $C = \{y\}$ we usually write $f^{-1}(y)$ instead of $f^{-1}(\{y\})$

The set $f^{-1}(C)$ is called the inverse image of C under f.

If $D \subset A$, then $f(D)$ is defined as $\{f(x) / x \in D\}$. The set $f(D)$ is called the image of D under f.

Example:

The set $f = \{(x, x^2) / -\infty < x < \infty\}$ in the function usually described by the equation $f(x) = x^2$ $(-\infty < x < \infty)$

the domain of this f is the whole real line. the range of f is $(0, \infty)$

Definition:

If f is a function from A into B, we write $f: A \to B$. If the range of f is all to B, we say that f is a function from A onto B. In this case we write $f: A \Rightarrow B$.

Thus, $f: (-\infty, \infty) \to (-\infty, \infty)$

$g: (-\infty, \infty) \Rightarrow (-\infty, \infty)$

Theorem: 2

If $f: A \to B$ and if $X \subset B$, $Y \subset B$, then

$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

(or)

The inverse image of the union of two sets is the union of the inverse images.

Proof:

Suppose $a \in f^{-1}(X \cup Y)$

then $f(a) \in X \cup Y$

Hence either $f(a) \in X$ or $f(a) \in Y$

$\Rightarrow a \in f^{-1}(X)$ or $a \in f^{-1}(Y)$

$\Rightarrow a \in f^{-1}(X) \cup f^{-1}(Y)$

$\therefore f^{-1}(X \cup Y) \subset f^{-1}(X) \cup f^{-1}(Y)$ ---------→(1)

conversely.

if $b \in f^{-1}(X) \cup f^{-1}(Y)$

then either $b \in f^{-1}(X)$ or $b \in f^{-1}(Y)$

$\Rightarrow$ either $f(b) \in X$ or $f(b) \in Y$

$\Rightarrow f(b) \in X \cup Y$

$\Rightarrow b \in f^{-1}(X \cup Y)$

Hence $f^{-1}(X) \cup f^{-1}(Y) \subset f^{-1}(X \cup Y)$

from (1) & (2),

$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

Hence proved.

THEOREM : 3

If $f : A \to B$ and $X \subset A$, $Y \subset A$ then

$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

The inverse image of the intersection of two sets is the intersection of the inverse images.

THEOREM: 4

If $f: A \to B$ and $X \subset A$, $Y \subset A$ then

$$f(X \cup Y) = f(X) \cup f(Y)$$

(or)

The image of the union of two sets is the union of the images

Proof:

If $b \in f(X \cup Y)$ then,

$b \in f(a)$ for some $a \in X \cup Y$

Either $a \in X$ or $a \in Y$

Thus either $b \in f(X)$ or $b \in f(Y)$

Hence $b \in f(X) \cup f(Y)$

$\therefore f(X \cup Y) \subset f(X) \cup f(Y)$

Conversely, if $c \in f(X) \cup f(Y)$

Then either $c \in f(X)$ or $c \in f(Y)$

Then c is the image of some point in X.

c is the image of some point in Y.

Hence c is the image of some point in $X \cup Y$,

i.e.) $c \in f(X \cup Y)$.

So $f(X) \cup f(Y) \subset f(X \cup Y)$

Hence $f(X \cup Y) = f(X) \cup f(Y)$

Note:

$f(A \cap B)$ need not be equal to

$f(A) \cap f(B)$.

the composition of functions.

If $f: A \to B$ and $g: B \to C$, then we

define the function $g \circ f$ by

$$g \circ f(x) = g[f(x)] \quad (x \in A)$$

(i.e) the image of x under $g \circ f$ is defined

to be the image of $f(x)$ under g. the function

$g \circ f$ is called the composition of f with g.

[write $g(f)$ instead of $g \circ f$]

Thus $g \circ f: A \to C$. For example, if

$$f(x) = 1 + \sin x \quad (-\infty < x < \infty),$$

$$g(x) = x^2 \quad (0 \leq x < \infty),$$

then $g \circ f(x) = 1 + 2\sin x + \sin^2 x \quad (-\infty < x < \infty)$

$$g[f(x)] = g[1 + \sin x]$$

$$= (1 + \sin x)^2$$

Real-valued functions 1.4:

If $f: A \to R$ we called f a real-valued function. If $x \in A$, then $f(x)$ [called the image of x under f] is also called the value of f at x.

Now define the sum, difference, product and quotient of real-valued functions.

If $f: A \to R$ and $g: A \to R$, we define $f+g$ as the function whose value at $x \in A$ is equal to $f(x) + g(x)$.

$$(i.e)(f+g)(x) = f(x) + g(x) \quad (x \in A)$$

In set notation,

$$f+g = \{(x, f(x)+g(x)) \mid x \in A\}$$

It is clear that $f+g : A \to R$.

$|||^{ly}$ we define $f-g$ and fg by

$$(f-g)(x) = f(x) - g(x) \quad (x \in A)$$

$$(fg)(x) = f(x)g(x) \quad x \in A$$

Finally, if $g(x) \neq 0 \ \forall x \in A$,

we define f/g by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad (x \in A)$$

The sum, difference, product and quotient of two real-valued functions with the same domain are again real valued functions.

Definition:

If $f: A \to R$ and c is a real number ($c \in R$), the function cf is defined by

$$(cf)(x) = c[f(x)] \quad (x \in A)$$

Thus the value of $3f$ at x is 3 times the value of f at x.

Definition:

If $f: A \to R$, $g: A \to R$ the $\max(f, g)(x) = \max[f(x), g(x)]$ $x \in A$ and

$\min(f, g)(x) = \min[f(x), g(x)]$ $(x \in A)$

$\therefore$ If $f: A \to R$ then $|f|$ is the function defined by $|f|(x) = |f(x)|$, $(x \in A)$

For real numbers a, b we have the formula.

$$\max(a, b) = \frac{|a-b| + a+b}{2}$$

$$\min(a, b) = \frac{-|a-b| + a+b}{2}$$

we get, $\max(f, g) = \dfrac{|f-g| + f+g}{2}$

$$\min(f, g) = \frac{-|f-g| + f+g}{2}$$

where f and g are real valued functions

Defn:

If $A \subset S$, then ψ_A [called the characteristic function of A] is defines as

$$\psi_A(x) = 1 \quad (x \in A)$$

$$\psi_A(x) = 0 \quad (x \notin A)$$

Proposition:

If A and B are two subsets of S than the following equations can easily be verified:

1. $\psi_{A \cup B} = \max(\psi_A, \psi_B)$

2. $\psi_{A \cap B} = \min(\psi_A, \psi_B) = \psi_A \psi_B$

3. $\psi_{A-B} = \psi_A - \psi_B$ (provided $B \subset A$)

4. $\psi_{A'} = 1 - \psi_A$

5. $\psi_\phi = 0$

6. $\psi_S = 1$

Equivalence, countability 1.5 :

Two sets A and B are said to be equivalent if there exists a bijective map $f: A \to B$.

Defn: 154

The set A is said to be countable or denumerable if A is equivalent to the set I of positive integers. An uncountable set is an infinite set which is not countable.

Thus A is countable set if f. a 1-1 function f from I onto A. The elements of A are then the images. $f(1), f(2) \ldots\ldots$ of positive integers.

Example: 1:

Let $A = \{ 1/2, \frac{2}{3}, \frac{3}{4} \ldots\ldots \}$. Then A is countable.

Since the mapping $f: I \to A: f(x^n) = \frac{n}{n+1}$ $\forall n \in I$ is bijective.

Hence A is countable.

Example: 2 155

The set of all integers is Countable.

Sol: Let $Z = \{0, -1, 1, -2, 2, \ldots\ldots\}$

Let $f: I \to Z$

$$f(n) = \frac{n-1}{2} \quad (n = 1, 3, 5, \ldots\ldots)$$

$$= \frac{-n}{2} \quad (n = 2, 4, 6)$$

Then f is clearly a bijective map.

Hence the set Z is countable.

Theorem : 5

A subset of a countable set is countable.

Proof:

Let A be any countable set and let $B \subset A$.

If B is finite, there is nothing to prove.

If B is infinite, then A is countably infinite. Then we can write A as an infinite sequence $\{a_1, a_2, \ldots\ldots a_n\}$.

Let n_1 be the least positive integer $\ni$: $a_{n_1} \in B$

Again let n_2 be the least (+ve) integer greater than $n_1 \ni$: $a_{n_2} \in B$ and so on.

Then it is evident that the mapping $f: I \to B$: $f(k) = a_{n_k}$ is bijective.

Hence B is countable.

Theorem : 6 1 S.T 10m

If $A_1, A_2, \ldots\ldots A_n$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable. (i.e) the countable union of countable set is countable.

Proof:

Let the countable sets $A_1, A_2, A_3, \ldots\ldots$ may be taken as

$A_1 = \{a_1^1, a_2^1, a_3^1, \ldots\ldots\}$

$A_2 = \{a_1^2, a_2^2, a_3^2, \ldots\ldots\}$

$\vdots$

$A_n = \{a_1^n, a_2^n, a_3^n, \ldots\ldots\}$

So that a_k^j is the k^{th} element of the set A_j

Let us define the height of a_k^j to be $j+k$.

Then a_1^1 is the only element of height 2.

III^{ly} a_2^1, a_1^2 are the only element of height 3 and so on.

Since for any the integer $m \geq 2$, there are only $m-1$ elements of height m.

We may arrange (count) the element of $\bigcup_{n=1}^{\infty} A_n$ according to their heights as.

$a_1^1, a_1^2, a_2^1, a_3^1, a_2^2, a_1^3, \ldots\ldots\ldots$

where we should remove any a_k^j that has already been counted.

The elements of $\bigcup_{n=1}^{\infty} A_n$ can be listed in the following Array.

$$\begin{array}{cccc}
a_1^1 & a_2^1 \rightarrow & a_3^1 & a_4^1 \\
\downarrow & & & \\
a_1^2 & a_2^2 & a_3^2 & a_4^2 \\
a_1^3 & a_2^3 & a_3^3 & a_4^2 \\
\downarrow & & & \\
a_1^4 & a_2^4 & a_3^4 & a_4^4
\end{array}$$

Thus counting scheme counts every a^j_k.

Hence $\bigcup_{n=1}^{\infty} A_n$ is countable.

Corollary : 1

The set of rational number is countable.

Proof:

Let $A_n = \left\{ \frac{0}{n}, \frac{-1}{n}, \frac{1}{n}, \frac{-2}{n}, \frac{2}{n}, \dots \right\}$

So that A_n is the set of all these rationals whose denominator is n.

Evidently each A_n is equivalent to the set of all integers.

$\therefore$ the function $f: I \to A_n$ defined by $f(r) = \frac{r-1}{2n}$ where r is odd.

$f(r) = -\frac{r}{2n}$ when r is even is one to one and onto.

Then each A_n is countable.

$\therefore$ Hence the set of all rationals is the union of $\bigcup_{n=1}^{\infty} A_n$ is countable.

Theorem : 7

The set of all rational numbers in $[0,1]$ is countable.

Proof: Prove the theorem (6)

We know that the set of all rational numbers is countable.

$\therefore$ The set of all rational $[0,1]$ is an infinite subset of rational. by thm.

$\therefore$ The set of rational in $[0,1]$ is countable.

Hence Proved.

1.6 Real Numbers:

We shall assume that every real number x can be written in decimal expansion.

$$x = b \cdot a_1 a_2 a_3 \ldots\ldots$$

$$= b + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots\ldots$$

where a_i's are integers $\ni : 0 \le a_i \le 9$.

This expansion is unique except for cases such as $x = 1/2$ which can expanded in two different forms

$$\frac{1}{2} = 0.5000\ldots\ldots \text{ and } 1/2 = 0.4999$$

Every number $x \in [0,1]$ can be expanded.

$x = 0 \cdot a_1 a_2 a_3 \cdots\cdots$

Conversely we assume that every decimal is of the form $b \cdot a_1, a_2, a_3 \cdots\cdots$ is the decimal expansion for same real number.

Theorem: 8 8. 5 mark

The set $[0,1] = \{x : 0 \le x \le 1\}$ is uncountable.

Proof:

Suppose $[0,1]$ is countable then we can write $[0,1] = \{a_1, a_2, a_3, \cdots\}$ where every number in $[0,1]$ occurs among the a_j.. $(j \in I)$

Expanding, each a_i in decimals we have,

$a_1 = 0 \cdot a^1_1 a^1_2 a^1_3 \cdots\cdots$

$a_2 = 0 \cdot a^2_1 a^2_2 a^2_3 \cdots\cdots$

$\vdots$

$a_n = 0 \cdot a^n_1 a^n_2 a^n_3 \cdots\cdots$

We now construct the real number $x = 0 \cdot x_1 x_2 x_3 \cdots\cdots x_n \cdots$ where x_1 is any integer form 1 to 8 $\ni$: $x_1 \neq a^1_1$

x_2 is any integer from 1 to 8 $\ni$: $x_2 \neq a^2_2$.

Ingeneral, x_n is any integer from 1 to 8 $\ni$: $x_n \neq a^n_n$

Then for any n the decimal expansion of x differs from the decimal expansion of a_n. Since $x_n \neq a_n^n$.

This decimal expansion of x is unique since no x_n is 0 or 9.

It follows that, x is different from a_n for all n and $0 \le x \le 1$.

But this contradicts the assumption that every number in $[0, 1]$ occurs among the a_i, $i \in I$.

$\therefore$ The set $[0, 1]$ is uncountable.

Hence Proved.

Corollary:

The set R of all real numbers is uncountable.

Proof:

We have already proved that a subset of a countable set is countable.

Hence if R were countable then $[0, 1]$ would also be countable contradicting the above theorem (8).

$\therefore$ Hence R is uncountable.

Ternary representations:

The digits 0, 1, 2 are called ternary digits.

If the series,

$$\frac{t_1}{3} + \frac{t_2}{3^2} + \frac{t_3}{3^3} + \cdots\cdots + \frac{t_n}{3^n} + \cdots\cdots, \text{ where } t_i = 0 \text{ or } 1 \text{ or } 2.$$

converges to x, then a ternary representation of x is defined by $x = 0_3 \cdot t_1 t_2 t_3 \cdots\cdots$ where the small 3 placed below the ternary point to emphasize that it is the ternary point.

For example,

$$\because \frac{2}{3} + \frac{0}{3^2} + \frac{2}{3^3} + \frac{0}{3^4} + \frac{2}{3^5} + \cdots\cdots$$

$$= \frac{2/3}{1-1/9} \rightarrow \frac{2/3}{8/9}$$

$$= \frac{2}{3} \times \frac{9}{8}$$

$$= 3/4$$

$$3/4 = 0_3 \cdot 202020 \cdots\cdots$$

Similarly

$$\frac{1}{3} = 0_3 \cdot 1000 \cdots\cdots$$

$$\frac{1}{3} = 0_3 \cdot 0222$$

$$\frac{1}{2} = 0_3 \cdot 1111 \cdots\cdots$$

Thus we find that the ternary expansion for a real number x is unique for numbers such as $1/4$ and $3/4$ and that some numbers such as $1/3$ are capable of two representations

Binary Representations.

Binary representation of x is given by

$x = 0_2 \cdot b_1 b_2 b_3 \ldots$ where each b_i is either 0 or 1

and where the series

$$\frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \cdots$$ converges to x.

For example,

$$\frac{1}{2} = 0_2 \cdot 1000 \ldots$$

$$\frac{1}{4} = 0_2 \cdot 01000 \ldots$$

$$\frac{1}{3} = 0_2 \cdot 010101 \ldots$$

Cantor Set:

The Cantor set K is the set of all numbers x in $[0, 1]$ which have ternary expansion without digit 1.

Thus the numbers $\frac{1}{3} = 0_3 \cdot 0222 \ldots$ and $\frac{2}{3} = 0_3 \cdot 2000 \ldots$ are in K but any number $x \ni : 1/3 < x < \frac{2}{3}$ are not in K.

theoram : 9 9^{th}

The cantor set K is uncountable.

Proof:

For any $x \in K$ we have

$x = 0_3 \cdot b_1 b_2 b_3 \ldots\ldots$ where each b_i is 0 or 2

Let $f(x) = y = 0_2 \cdot a_1 a_2 a_3 \ldots\ldots$

where $a_i = \frac{b_i}{2}$

then $0 \leq y \leq 1$ and f is a function from K into $[0,1]$. In fact it is onto $[0,1]$.

By a theorem,

"If $f: A \to B$ and the range of f is uncountable. then domain of f is uncountable.

$\therefore$ K is uncountable as its range $[0,1]$ is uncountable.

The cantor set K is obtained in the following way:

1. From $[0,1]$ remove the open middle third leaving $[0, 1/3]$ and $[1/2, 2/3]$.

2. From each of $[0, 1/3]$ and $[1/3, 2/3]$ remove the open middle third leaving.

$[0, 1/9], [2/9, 3/9], [6/9, 7/9], [8/9, 9/9]$.

This process is continued so that at n^{th} step the open middle third is removed from each of 2^{n-1} intervals & of length 3^{-n+1}.

the total lengths removed at the n^{th} step is $2^{n-1} \cdot 1/3 \cdot 3^{-n+1} = \frac{2^{n-1}}{3^n}$

Then there remains 2^n intervals each of length 3^{-n}.

It is clear that, what remains of $[0,1]$ after this process is continued indefinitely is the set K.

We also that the sum of lengths at the intervals in K' is $1/3 + 2/9 + \cdots + \frac{2^{n-1}}{3^n} + \cdots$

Thus $K \subset [0,1]$ and is the complement of the intervals whose lengths add upto 1.

Problems:

1. Prove that any infinite set contains a countable subset.

Sol: Let A be an infinite set.

Let us choose arbitrarily an element a_1 of A.

Since A is infinite, we can choose another element a_2 from the remaining $A - \{a_1\}$.

Let us next choose a_3 from $A - \{a_1, a_2\}$ we can continue this process of choosing the elements indefinitely.

As a result we obtain a subset $\{a_1, a_2, a_3, \ldots\ldots a_n\}$ of A which is clearly countable.

Example : 2

the set $I \times I$ is countable.

Soln: Consider the subset

$M = \{2^m 3^n \in I : (m, n) \in I \times I\}$ of I

Since I is countable and $M \subset I$. it follows that M is countable and.

$\therefore f$ is a surjective function $f : I \to M$.

We now define a function $g : M \to I \times I$ by $g(2^m 3^n) = (m, n)$.

Evidently g is surjective. ~~function f:I→M~~

the composition function.

$(g \circ f)(n) = g[f(n)], \forall n \in I$ is then a surjective function.

Hence $I \times I$ is countable.

Example : 3

If A and B are countable sets, then the cartesian product $A \times B$ is ~~a~~ countable.

Let $A = \{a_1, a_2, a_3, \ldots\ldots\}$

$B = \{b_1, b_2, b_3, \ldots\ldots\}$ and

let $A_n = A \times \{b_n\}$

$= \{(a, b_n) ; a \in A\}$

Then A_n is countable for each $n=1,2,3,\ldots$

and $A \times B = \bigcup_{n=1}^{\infty} A_n$.

Hence $A \times B$ is countable. (the countable union of countable set is countable).

Example : 4

Every infinite set is equivalent to a proper subset.

soln: Let A be an infinite set.

Hence it contains a denumerable subset, say $\{a_1, a_2, \ldots\}$

Let A^* be the subset of A after removing the elements of this denumerable subset from A so that,

$$A = \{a_1, a_2, a_3 \ldots\} \cup A^*$$

Now consider the mapping $f : A \to A - \{a_1\}$ defined by $f(a_n) = a_{n+1}$, $n = 1, 2, 3, \ldots$ and $f(a) = a \ \forall \ a \in A^*$

the mapping f is clearly one-one, onto

$\Rightarrow A$ is equivalent to proper subset $A - \{a_1\}$.

Hence proved.

Example 5:

Show that if B is a countable subset of an uncountable set A then A-B is uncountable.

Sol: Proof:

If A-B were countable then $(A-B)\cup B$ be countable, being the union of two countable sets.

But since $B \subset A$ we have,

$$(A-B)\cup B = A$$

Thus A would be countable.

which is $\Rightarrow\Leftarrow$.

Hence we conclude that,

A-B would be uncountable.

Example: 6

Show that the set of all irrational numbers is uncountable.

Sol:

$\because$ the set R of real numbers is uncountable and the set Q of rational numbers is countable it follows from the above example that the set R-Q of all irrational number is uncountable.

1.7. Least upper bounds

Defn: The subset $A \subset R$ is said to be bounded above if there is a number $N \in R$ such that $x \leq N$ for every $x \in A$.

The subset $A \subset R$ is said to be bounded below if there is a number $M \in R$ such that $M \leq x$ for every $x \in A$.

If A is both bounded below and bounded above, we say that A is bounded.

Examples of sets bounded above:

1. Set of negative integers.
2. $A = \{x : 0 < x < 1\}$
3. The closed interval $[0,1]$

Examples of bounded sets below:

1. Set of positive integers
2. The closed interval $[0,1]$
3. Let $E = [0, \infty]$. Then E is bounded below by 0 but not bounded above.

Examples of bounded set:

1. Let $E = \{1, 1/2, 1/3, \ldots\}$ then 1 is an upper bound and 0 is a

lower bound for E. Hence E is bounded.

2. The closed interval $[0, 1]$ is bounded above by 1 and is bounded below by 0.

Definition:

Let the subset S of R be bounded above. The number L is called the least upper bound (l.u.b) for S if

(1) L is an upper bound for S.

2.) No number smaller than L is an upper bound for S.

$|||^{ly}$ l is called the greatest lower bound (g.l.b) for the set S bounded below if l is a lower bound for S and no number greater than l is a lower bound for S.

Examples:

1. For $S = [0,1]$, l.u.b is 1 and g.l.b is 0.

2. For $S = \{ 1/2, 3/4, \ldots \frac{2^n-1}{2^n}, \ldots \}$

l.u.b = 1 and g.l.b = 1/2

[Note: l.u.b of S is not an elt of S]

3. For the set N of all integers g.l.b is 1 and there is no l.u.b

Since N is not bounded above.

4. For $S = (7, 8)$ open interval does not contain either l.u.b or to g.l.b which are 7 and 8 respectively.

5. For the singleton set $\{5\}$

$l.u.b = g.l.b = 5$

The least upper bound axiom:

(or)

The completeness axiom:

Every non-empty set S of real numbers which is bounded above has a l.u.b, there is a real number $c \ni$:

$c = l.u.b$ of S.

Theorem 1.10

Every non-empty set S that is bounded below has a g.l.b

Proof:

Let $A = \{x : x \in R, -x \in S\}$

$\because S$ is bounded below, let l be a lower bound of S.

Now $x \in A \Rightarrow -x \in S$

$l \leq -x$

($\because l$ lower bound of S

$\Rightarrow +x \le -1$ $\quad -1 \ge x$
$x \le -1$

$\Rightarrow -1$ is an upper bound of A

Thus A is an non-empty set of real numbers which is bounded above and so by l.u.b axiom, A has a l.u.b.

If 'a' is the l.u.b for A then

$-a$ is the g.l.b for S.

For $a =$ l.u.b A.

$\Rightarrow a$ is an u.b for A

$\Rightarrow -a$ is l.b for S.

Also $x > -a \Rightarrow -x < a$.

$-x$ is not an u.b for A.

x is not a l.b for S.

Thus $-a =$ g.l.b S

2. Sequences of real numbers:

2.1 Definition of sequence and subsequence:

* A sequence $s = \{s_i\}_{i=1}^{\infty}$ of real numbers is a function from I (the set of +ve integers) into R (the set of real numbers).

The number s_i $(i = 1, 2, 3 \ldots)$ is called ith term of the sequence.

If $s_1, s_2, s_3 \ldots$ is a sequence, a subsequence is usually written as $s_{n_1}, s_{n_2}, s_{n_3}$.

* A Subsequence N of $\{n\}_{n=1}^{\infty}$ is a function from I (the set of +ve integers) into I ∋:

$N(i) < N(j)$ if $i < j$ $(i, j \in I)$

A Subsequence of $\{n\}_{n=1}^{\infty}$ is a sequence of integers whose terms become larger and larger.

Examples:

The sequence of primes 2, 3, 5, 7, 11, 13, ... is a subsequence of $\{n\}$

The even number 2, 4, 6, 8 ...... and the odd numbers 1, 3, 5, 7 .... are also subsequences of $\{n\}$

Definition:

If $S = \{S_n\}_{n=1}^{\infty}$ is a Sequence of real numbers and $N = \{n_i\}_{i=1}^{\infty}$ is a Subsequence of +ve integers then the composite functions SoN is called a Subsequence of S.

Note:

For $i \in I$ we have $N(i) = n_i$

$\therefore SoN = S[N(i)] = S(n_i) = S_{n_i}$

Examples: $S \circ N = \{S_{n_i}\}_{i=1}^{\infty}$

1. Let B denote the sequence $1, 0, 1, 0, \ldots\ldots$

Define $N = \{n_i\}_{i=1}^{\infty}$ by $n_i = 2i-1$ $(i \in I)$

So that $n_1 = 2-1 = 1$, $n_2 = 4-1 = 3$, $n_3 = 5 \ldots\ldots$

Then $B \circ N = \{B_{n_i}\}_{i=1}^{\infty}$ is $1, 1, 1, \ldots\ldots$ which is a

Subsequence of B. $B(N(i)) = 1, 1, 1, \ldots$

2. Let $C = \{C_n\}_{n=1}^{\infty} = \{\sqrt{n}\}_{n=1}^{\infty}$ and

$N = \{n_i\}_{i=1}^{\infty} = \{i^4\}_{i=1}^{\infty}$ $C\{N(i)\} = C n_i$

Then, $C \circ N = \{C_{n_i}\}_{i=1}^{\infty} = \{\sqrt{i^4}\} = \{i^2\}_{i=1}^{\infty}$ is a

Subsequence of C.

2.2 Limit of a Sequence:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that S_n approches the limit L as n approches infinity if for every $\epsilon > 0$ there is a positive integer N $\ni$: $|S_n - L| < \epsilon$ $(n \geq N)$

If S_n approaches the limit L we write

$\lim\limits_{n \to \infty} S_n = L$ or $S_{n_i} \to L$.

Sol:-

Given $\epsilon > 0$, we have to find a positive integer $N \ni : |S_n - 3| < \epsilon$

(i.e) $\left| \frac{3n}{n+5\sqrt{n}} - 3 \right| < \epsilon \quad \forall \quad n \geq N$ -------> (1)

(i.e) $\left| \frac{3n-3n-15\sqrt{n}}{n+5\sqrt{n}} \right| < \epsilon \quad (n \geq N)$

$\frac{15\sqrt{n}}{n+5\sqrt{n}} < \epsilon \quad (n \geq N)$ ----------> (2)

But, $\frac{15\sqrt{n}}{n+5\sqrt{n}} < \frac{15\sqrt{n}}{n} = \frac{15}{\sqrt{n}}$

$\therefore$ (2) will hold if $\frac{15}{\sqrt{n}} < \epsilon$ ----------> (3)

If we choose N so large that $\frac{15}{\sqrt{n}} < \epsilon$

(i.e) choose $N > \frac{225}{\epsilon^2}$ then (3) will certainly hold and consequently (1) will hold for any positive integers

$N > \frac{225}{\epsilon^2}$

$\therefore \lim_{n \to \infty} S_n = 3$

Example 4:

Consider $\{S_n\}_{n=1}^{\infty}$ where $S_n = n$ ($i = 1, 2, 3 \ldots\ldots$)

Prove that this sequence does not have a limit

Sol:-

Assume the contrary, then

$\lim_{n \to \infty} S_n = L$ for some $L \in R$.

(i.e) given $\epsilon > 0$ there is an $N \in I$ s:

$|S_n - L| < \epsilon \quad (n \geq N)$

In particular for $\epsilon = 1$, we have,

$|S_n - L| < 1 \quad (n \geq N)$

(i.e) $-1 < S_n - L < 1 \quad (n \geq N)$

$L - 1 < S_n < L + 1$ for $n \geq N$.

$L - 1 < n < L + 1$ (given $S_n = n$)

All values of n greater than N line between $L-1$ and $L+1$ which is clearly false and the contradiction shows that $\{S_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ does not have a limit.

Example 5:

Show that sequence $\{S_n\}_{n=1}^{\infty}$ where $S_n = (-1)^n \quad (n = 1, 2, 3, \ldots)$ does not converge.

Soln:

Let us assume that there is a $L \in R$ for which $\lim_{n \to \infty} S_n = L$.

Taking $\epsilon = 1/2$, there must be an

$|S_n - L| < 1/2 \quad (n \geq N)$

(i.e) $|(-1)^n - L| < 1/2 \quad (n \geq N)$ -------→ (1)

For n even (1) becomes

$|1-l| < 1/2$ and

for n odd,

$$|-(1+l)| < 1/2 \quad \text{-------} \rightarrow (2)$$

Now $\quad 2 = |2| = |1+1|$

$$= |(1+l)+(1-l)|$$

$$= |1+l| + |1-l| < 1/2 + 1/2 = 1$$

which is a $\Rightarrow\Leftarrow$

Hence no limit l exists for the sequence

$\{(-1)^n\}_{n=1}^{\infty}$

Example : 6

Show that the sequence $\{S_n\}$ where

$S_n = \dfrac{n^2+1}{2n^2+5}$ $\forall\ n \in I$ converges to '$1/2$'.

Soln: Given $\epsilon > 0$, we have to find a (+ve)

integer $N \ni$:

$$\left|\frac{n^2+1}{2n^2+5} - \frac{1}{2}\right| < \epsilon \quad \forall n \geq N \quad \text{---------} \rightarrow (1)$$

(i.e) $\left|\dfrac{2n^2+2-2n^2-5}{4n^2+10}\right| < \epsilon \quad \forall\ n \geq N.$

(i.e) $\left|\dfrac{-3}{4n^2+10}\right| < \epsilon \quad \forall\ n \geq N$ (i.e) $\dfrac{3}{2(2n^2+5)} < \epsilon\ \forall n \geq 5$ ------$\rightarrow(2)$

But $\dfrac{3}{2(2n^2+5)} < \dfrac{3}{4n^2}$

$\therefore$ (2) will hold if $\frac{3}{4n^2} < \epsilon$ ---------->(3)

or $n^2 > \frac{3}{4\epsilon}$ (i.e) $n > \sqrt{\frac{3}{4\epsilon}}$

If we choose N so large that $N > \sqrt{\frac{3}{4\epsilon}}$ then

(3) will certainly hold and consequently (1)

will hold for any positive integer $N > \sqrt{\frac{3}{4\epsilon}}$

$$\therefore \lim_{n\to\infty} S_n = 1/2$$

Example: 7

Show that $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$ converges to 0.

Sol: Let $S_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1}) + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \Rightarrow \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$$

$$= \frac{1}{2\sqrt{n}}$$

Given $\epsilon > 0$ we must find a (+ve) integer

N $\ni$: $|S_n - 0| < \epsilon$ (i.e) $S_n < \epsilon$

(i.e) $\frac{1}{2\sqrt{n}} < \epsilon$ (or) $n > \frac{1}{4\epsilon^2}$

If we choose N so large that $N > \frac{1}{4\epsilon^2}$

then $|s_n - 0| < \epsilon \quad (n \geq N)$

$\therefore \lim_{n\to\infty} s_n = 0$

(8) **Theorem: 11**

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of non-negative numbers and if $\lim_{n\to\infty} s_n = L$ then $L \geq 0$

Proof:

Suppose the contrary (ie) $L < 0$.

Then for $\epsilon > 0$ $\exists$ $N \in I \ni$:

$|s_n - L| < \epsilon \quad (n \geq N)$

Taking $\epsilon = -L/2$ and $n = N$, we have

$|s_N - L| < -L/2$ which implies

$s_N - L < -L/2$

(i.e) $s_N < L/2$

$\Rightarrow s_n < 0 \quad (\because L < 0)$

This is a condition to the hypothesis $s_N \geq 0$.

Hence $L \geq 0$

2nd

2.3 convergent sequences:

Defn: If the sequences of real numbers $\{s_n\}_{n=1}^{\infty}$ has a limit L, we say that $\{s_n\}_{n=1}^{\infty}$ is convergent to L. If $\{s_n\}_{n=1}^{\infty}$ does not have a limit, we say that $\{s_n\}_{n=1}^{\infty}$ is divergent.

Example:

(i) The sequence 1, 1, 1, 1 ...... converges to 1

(ii) The sequence 1, 1/2, 1/3 ...... converges to 0

(iii) The sequence 1, 2, 3, ... and 1, -1, 1, -1, 1, ...

are divergent.

Theorem: 12 [UNIQUENESS OF LIMIT]

Statement: (1.5)

If the sequence of real number $\{s_n\}_{n=1}^{\infty}$ is convergent to L then $\{s_n\}_{n=1}^{\infty}$ cannot also converge to a limit distinct from L. (i.e.) if $\lim_{n \to \infty} s_n = L$ and $\lim_{n \to \infty} s_n = M$ then $L = M$

Proof:

Assume the contrary.

then $L \neq M$. So that $|M - L| > 0$

$$\text{Let } \epsilon = \frac{1}{2} |M - L| \text{ ---------> (1)}$$

By hypothesis, $\lim_{n \to \infty} s_n = L$, $\exists N_1 \in I \ni$:

$$|s_n - L| < \epsilon \quad (n \geq N_1) \text{ ---------> (2)}$$

III^{ly} Since $\lim_{n \to \infty} s_n = M$, $\exists N_2 \in I \ni$:

$$|s_n - M| < \epsilon \quad (n \geq N_2) \text{ ---------> (3)}$$

Let $N = \text{Max}(N_1, N_2)$ then clearly

$N \geq N_1$ and $N \geq N_2$

$\therefore$ For such and N both (2) & (3) hold.

$$|(s_n - L) - (s_n - M)|$$

$\therefore |M-L| = |(S_n - L) - (S_n - M)|$

$\leq |(S_N - L)| + |(S_N - M)|$

$< \epsilon + \epsilon = 2\epsilon = |M-L|$ using (1)

(i.e) $|M-L| < |M-L|$ which is a $\Rightarrow\Leftarrow$

$\therefore M = L$.

Theorem: 13

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent to L, then any subsequence of $\{S_n\}_{n=1}^{\infty}$ is also convergent to L.

Proof:

Let $\{S_{n_k}\}$ be any subsequence of $\{S_n\}_{n=1}^{\infty}$

then by defn of a sequence,

$n_1, n_2, n_3 \ldots\ldots\ldots$ are positive integers $\ni$:

$n_1 < n_2 < n_3 < \ldots\ldots\ldots n_k < \ldots\ldots\ldots \rightarrow (1)$

Let $\epsilon > 0$ be given.

$\therefore \lim_{n\to\infty} S_n = L$, $\exists N \in I \ni : |S_n - L| < \epsilon \ (n \geq N) \rightarrow (2)$

By condition of (1), $\exists$ a value of k say $k_0 \ni$:

$n_{k_0} \geq N \rightarrow (3)$

Then (2) & (3) $\Rightarrow |(S_{n_k} - L)| < \epsilon \quad (n \geq n_{k_0})$

(i.e) the subsequence $\{S_{n_k}\}$ converges to L.

12 b) If $\{S_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers then prove that $\{S_n\}_{n=1}^{\infty}$ is bounded.

Corollary:

All subsequences of a convergente Sequence converge to the same limit.

Proof: By theorem 12, any subsequence of a sequence converges to the save limit of the sequence and by theorem 11, the limit of a sequence is unique.

$\Rightarrow$ All subsequences of a convergent sequences have the same limit.

Example:

Show that the sequence $\{S_n\}_{n=1}^{\infty}$ where $S_n = \sin n\pi\theta$ and θ is rational number $\ni : 0<\theta<1$ is not convergent.

soln: Let $\theta = p/q$ where p and q are integer since $0<\theta<1$, we have $q \geq 2$.

for $n = q, 2q, 3q, \ldots\ldots$ the terms of the given sequence are $\sin \pi p, \sin 2\pi p, \sin 3\pi p, \ldots\ldots$

(ie) $0, 0, 0, \ldots\ldots$

thus S contains a subsequence which converges to 0.

Again for $n = q+1, 2q+1, 3q+1, \ldots\ldots$ the terms of seq are

$\sin\left(\pi p+\frac{\pi p}{q}\right), \sin\left(2\pi p+\frac{\pi p}{q}\right), \sin\left(3\pi p+\frac{\pi p}{q}\right)\ldots\ldots$

(or)

$(-1)^{p}\sin\left(\frac{\pi p}{q}\right), (-1)^{2p}\sin\left(\frac{\pi p}{q}\right), (-1)^{3p}\sin\left(\frac{\pi p}{q}\right)\ldots\ldots$

These terms all have absolute value $\sin\frac{\pi p}{q}$ and do not approach zero.

Since $0 < \frac{\pi p}{q} < \pi$

Thus S contains a subsequence whose limit is 0 and a subsequence which may or may not converge but certainly does not have a limit 0.

Hence by corollary of theorem 13, the given sequence is not convergent.

For $\theta=0$ or $\theta=1$, the sequence $\{\sin n\theta\pi\}_{n=1}^{\infty}$ clearly converges to 0.

UNIT-II

SEQUENCES [CONTD....]

2.4 Divergent Sequences:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequences of real numbers we say that S_n approaches infinity as n approaches infinity if for any real number $M>0$ there is a positive integer $N \ni$: $S_n \geq M$ $(n \geq N)$. In this case we write $S_n \to \infty$ as $n \to \infty$.

Ex:

Show that the sequence $\{n\}_{n=1}^{\infty}$ diverges to infinity.

Soln: Let $S_n = n$

For given $M>0$, first choose $N \in I_+ \ni: N \geq M$

then certainly $n \geq M$ $(n \geq N)$

(ie) $S_n \geq M$ $(n \geq N)$ $\therefore S_n \to \infty$ as $n \to \infty$.

Defn: Let $\{S_n\}_{n=1}^{\infty}$ be a Sequence of real number we say that S_n approches minus infinity as n approaches infinity if, for any real number $M>0$, there is a +ve integer $N \ni$: $S_n < -M$. we write $S_n \to \infty$ as $n \to \infty$

Ex: Show that the sequence $\{S_n\}$ where $S_n = \log_e(1/n)$ diverges to $-\infty$

Sol: Let $M > 0$ be given. Then we must find $N \in I \ni$: $\log_e(1/n) < -M \quad (n \geq N)$ ----------→(1)

$\because \frac{1}{n} = n^{-1}$
$\log n^{-1} = -\log n$
$-\log n < -M$
$\log n > M$

(i.e) $-\log_e n < -M \quad (n \geq N)$

(or) $\log_e n > M \quad (n \geq N)$

$n \geq e^M \quad (n \geq N)$ ------→(2)

Thus if we choose $N \geq e^M$, then (2) and hence (1) will hold.

It follows that $\log(1/n) \to -\infty$

Theorem 1: 2.4.1

If a sequence S_n diverges to ∞ or $-\infty$ so does any subsequence of $\{S_n\}$

Sol: Proof is similar of theorem (2.3)

Defn: If the sequence $\{S_n\}_{n=1}^{\infty}$ of real numbers diverges but does not diverge to infinity and does not diverge to minus infinity, we say than $\{S_n\}_{n=1}^{\infty}$ oscillates.

Ex: (i) The sequence $\{(-1)^n\}_{n=1}^{\infty}$ oscillates $-1, 1, -1, \ldots\ldots$

(ii) The sequences $1, 2, 1, 3, 1, 4, 1, 5, \ldots\ldots\ldots$ is another example of oscillating sequences.

3rd

2.5 Bounded Sequences:

Defn: If the sequences $\{S_n\}_{n=1}^{\infty}$ is bounded

above if the range of $\{S_n\}_{n=1}^{\infty}$ is bounded

above. lll^ly, if the sequences $\{S_n\}_{n=1}^{\infty}$ is bounded

below. A sequence is said to be bounded if it

is both bounded above and below.

thus $\{S_n\}_{n=1}^{\infty}$ is bounded iff $\exists$ $M \in R \ni$;

$|S_n| \leq M$; $(n \in I)$

Note:

1. If a sequences diverges to infinity (or to minus infinity) the sequences is not bounded.

2. A sequences that diverges to infinity must, however be bounded below. An oscillating sequences may or may not be bounded.

Ex:

1. the oscillating sequences $\{(-1)^n\}_{n=1}^{\infty}$ is bounded since its range set is $\{-1, 1\}$ which is bounded.

2. the oscillating sequence $1, -2, 3, -4, \ldots\ldots$ is not bounded. It is neither bounded below or bounded above.

3. The sequence $1, 2, 1, 3, 1, 4, \ldots\ldots$ oscillating and is bounded below but is not bounded above.

However, a convergent sequences is always bounded.

theorem : 2.5.1

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ converges, then $\{S_n\}_{n=1}^{\infty}$ is bounded.

Proof:

Since $\{S_n\}_{n=1}^{\infty}$ is convergent, it has a limit.

Let $\lim_{n \to \infty} S_n = L$

then given, $\epsilon = 1$, $\exists N \in I \ni$:

$$|S_n - L| < 1 \quad (n \geq N)$$

Now,
$$\begin{aligned} |S_n| &= |(S_n - L) + L| \\ &\leq |(S_n - L)| + |L| \\ &\leq 1 + |L| \quad (n \geq N) \quad \text{------} (1) \end{aligned}$$

If we let $M = \max\{|S_1|, |S_2|, \ldots\ldots |S_{N-1}|\}$ then, we have $|S_n| < M + |L| + 1$ $(n \in I)$

which shows that $\{S_n\}_{n=1}^{\infty}$ is bounded.

2.6 monotone sequences:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $S_1 \leq S_2 \leq S_3 \leq \ldots\ldots \leq S_n \leq S_{n+1} \leq \ldots\ldots$, then $\{S_n\}_{n=1}^{\infty}$ is called non-increasing.

A monotone sequence is a sequence which is either non-Increasing or non-decreasing.

Example:

1. The sequence $\{1/2, 2/3, 3/4, \ldots \frac{n}{n+1}, \ldots\}$ is non-decreasing and bounded.

2. The sequence $\{2, 3/2, 4/3, 5/4, \ldots \frac{n}{n+1}, \ldots\}$ is non-~~decreas~~ increasing and bounded.

3. The sequence $\{1, |1/2|, |3/4|, |7/8|, \ldots 2-\frac{1}{2^{n-1}}, \ldots\}$ is non-decreasing and bounded.

4. The sequence $\{n\}_{n=1}^{\infty}$ is non-decreasing and not bounded.

5. The sequence $\{1, -1/2, 1/3, -1/4, \ldots\}$ is not monotone but bounded.

Theorem: 2.6.1

A non-decreasing sequence which is bounded above is convergent.

Proof:

Suppose $\{S_n\}_{n=1}^{\infty}$ is non decreasing and bounded above.

Then the set $S = \{S_1, S_2, \ldots\}$ is a non empty subset of R which is bounded above.

By the least acceptance axiom for R, S must be have l.u.b

Let $M = \text{l.u.b}\{s_1, s_2, \ldots\} = \text{l.u.b}$ for S. we

Shall prove $s_n \to M$ as $n \to \infty$

Given $\epsilon > 0$, the number $M - \epsilon$ is not an

upper bounded for S

Hence for some $N \in I$, $s_N > M - \epsilon$.

But since $\{s_n\}_{n=1}^{\infty}$ is non-decreasing

$s_n > M - \epsilon \quad (n \geq N)$ ....(1)

on the other hand, since M is an upper bound for S.

$s_n \leq M \quad (n \in I)$ ....(2)

From (1) & (2) we conclude that

$M - \epsilon < s_n < M$

$\Rightarrow |s_n - M| < \epsilon \quad (n \geq N)$

$\lim_{n \to \infty} s_n = M$ (i.e) $\{s_n\}$ converges to M.

Note:

Least upper bound axiom:

Any non empty subset of the real line R which is bounded above has a least upper bound.

Example: The sequence $\{s_n\}_{n=1}^{\infty}$ where $s_n = (1 + 1/n)^n$ is convergent

sol: Let $s_n = (1 + 1/n)^n$

$$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{1 \cdot 2} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{n^3} + \cdots + \frac{n(n-1) \cdots 1}{1 \cdot 2 \cdots n} \frac{1}{n^n} \quad \cdots\cdots (1)$$

$$S_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= 1+(n+1)\frac{1}{(n+1)} + \frac{(n+1)n}{1\cdot 2}\frac{1}{(n+1)^2} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}\frac{1}{(n+1)^3}$$

$$\cdots\cdots + \frac{(n+1)n\cdots\cdots}{1\cdot 2\cdots\cdots(n+1)}\frac{1}{(n+1)^{n+1}} \cdots\cdots (2)$$

Let T_{r+1} and T'_{r+1} denote the $(r+1)^{th}$ terms in (1) & (2) respectively for $r=1,2,\ldots\ldots n$.

then $T_{r+1} = \dfrac{n(n-1)\cdots\cdots(n-r+1)}{1\cdot 2\cdots\cdots r}\dfrac{1}{n^r}$

$$= \frac{1}{1\cdot 2\cdots r}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{r-1}{n}\right)\right]$$

 $|||^{ly}$,

$$T'_{r+1} = \frac{1}{1\cdot 2\cdots r}\left[\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\cdots\left(1-\frac{r-1}{n+1}\right)\right]$$

we have, $\Rightarrow 1-\left(\dfrac{r-1}{n+1}\right) > 1-\left(\dfrac{r-1}{n}\right)$

for $r=1,2\ldots\ldots n$

thus $T_{r+1} < T'_{r+1}$ for $r=1,2,3\ldots\ldots$

$\therefore S_n \leq S_{n+1} \Rightarrow \{S_n\}$ is non-decreasing

Further,

$$S_n = 1+1+\frac{1}{1\cdot 2}\left(1-\frac{1}{n}\right)+\frac{1}{1\cdot 2\cdot 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\cdots$$

$$+\frac{1}{1\cdot 2\cdots n}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$< 1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots\cdots\frac{1}{1\cdot 2\cdots n}$$

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{n+1}}$$

$$< 1+\frac{1-(1/2)^n}{1-1/2} \Rightarrow 1+\frac{1-0}{1/2}$$

$$= 1+2/1$$

$$= 3$$

Hence $\{S_n\}_{n=1}^{\infty}$ is bounded above.

Hence by theorem (3) $\{S_n\}_{n=1}^{\infty}$ is convergent.

Note:

To denote $\lim_{n \to \infty} (1+1/n)^n$ by the symbol e.

The above example shows that $2 < e \leq 3$

Theorem : 4

A non-decreasing sequence which is not bounded above diverges to infinity.

Proof:

Suppose $\{S_n\}_{n=1}^{\infty}$ is non-decreasing but not bounded above.

Given $M > 0$, we must find $N \in I \ni$:

$$S_n > M, \ (n \geq N) \text{------} \to (1)$$

Since M is not an upper bound for $\{S_1, S_2, \ldots\}$ there must exist $N \in I$.

$\ni : S_N > M$

For this N, (1) follows from the hypothesis that $\{S_n\}_{n=1}^{\infty}$ is non-decreasing.

Hence Proved.

Theorem: 5

(a) A non-increasing sequence which is bounded below is convergent.

(b) A non-increasing sequence which is not bounded below diverges to minus infinity.

Proof: The proof of the above theorem follows the proofs of theorems 3 and 4 exactly with the all upper bounds and least upper bounds replaced by lower bound and greatest lower bounds.

2.7 Operations on convergent sequences:

Theorem: 6

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n\to\infty} s_n = L$, and if $\lim_{n\to\infty} t_n = M$ then $\lim_{n\to\infty} (s_n + t_n) = L+M$

(or)

The limit of the sum of two convergent sequences is the sum of the limits.

Proof: Given $\epsilon > 0$ we must find $N \in I \ni$:

$|s_n + t_n - (L+M)| < \epsilon \quad (n \geq N)$ ---------->(1)

Since $\lim_{t\to\infty} s_n = L \ \exists\ N_1 \in I \ni : |s_n - L| < \epsilon/2 \ (n \geq N_1)$ ---->(2)

Also since $\lim_{t\to\infty} t_n = M, \exists N_2 \in I \ni : |t_n - M| < \epsilon/2 \ (n \geq N_2)$

Let $N = \max(N_1, N_2)$. Then (2) & (3) hold for all $n \geq N$.

Thus for all $n \geq N$

$$|(s_n + t_n) - (L+M)| = |(s_n - L) + (t_n - M)|$$
$$\leq |(s_n - L)| + |(t_n - M)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{[from (1) \& (2)]}$$

Hence $\lim_{n \to \infty} (s_n + t_n) = L + M$.

Theorem : 7

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers. If $c \in R$ and if $\lim_{n \to \infty} s_n = L$, then $\lim_{n \to \infty} (s_n \times c)$

Proof:

If $c = 0$, the theorem is true.

$\therefore$ Assume that $c \neq 0$.

Let $\epsilon > 0$ be given, we must find $N \in I \ni$:

$|cs_n - cL| < \epsilon \quad (n \geq N)$ ------> (1)

Since $\lim_{n \to \infty} s_n = L$, $\exists N \in I \ni$:

$|c|\,|s_n - L| < \epsilon \quad (n \geq N)$

This is equivalent to (1)

Hence $\lim_{n \to \infty} cs_n = cL$.

Theorem : 8

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers. If $\lim_{n \to \infty} s_n = L$ and $\lim_{n \to \infty} t_n = M$ then

$\lim_{n \to \infty} (s_n - t_n) = L - M$

Since $\lim_{n\to\infty} t_n = M$ it follows from theorem 7

(with $c=-1$) that.

$$\lim_{n\to\infty} (-t_n) = -M$$

Then by theorem (6)

$$\lim_{n\to\infty} (s_n - t_n) = \lim (s_n + (-t_n))$$

$$= \lim_{n\to\infty} s_n + \lim_{n\to\infty} (-t_n)$$

$$= L + (-M)$$

$$\lim_{n\to\infty} (s_n - t_n) = L - M.$$

Corollary:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers if $s_n \le t_n$ $(n \in I)$ and if $\lim_{n\to\infty} s_n = L$ $\lim_{n\to\infty} t_n = M$ then $L \le M$.

Soln: By theorem (8).

$$\lim_{n\to\infty} (t_n - s_n) = M - L$$

But $t_n - s_n \ge 0$ $(n \in I)$ [By theorem (11) in unit I]

$M - L \ge 0$ (i.e) $L \le M$

Lamma : 1

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers which converges to L, then $\{s_n^2\}_{n=1}^{\infty}$ converges to L^2.

Soln: Given $\epsilon > 0$, we must find $N \in I$ s:

$$|s_n^2 - L^2| < \epsilon \quad (n \ge N)$$

(i.e) $|s_n - L|\,|s_n + L| < \epsilon$ $(n \ge N)$ ---------> (1)

since $\{S_n\}_{n=1}^{\infty}$ converges by Theorem 2.5 it is bounded.

Thus for some $M > 0$ $|S_n| \le M$ $(n \in I)$ so that $|S_n + L| \le |S_n| + |L| \le M + |L|$ $(n \in I)$ ------> (2)

since $\lim_{n\to\infty} S_n = L$ $\exists$ $N \in I$ $\ni$

$$|S_n - L| < \frac{\epsilon}{M+|L|} \quad (n \ge N)$$

multiplying (2) & (3),

$$|S_n + L|\,|S_n - L| < \frac{\epsilon}{M+|L|}\,(M+|L|) = \epsilon$$

thus for this N, (1) holds

Hence $\lim_{n\to\infty} S_n^2 = L^2$

theorem 9:

If $\{S_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequence of real numbers, if $\lim_{n\to\infty} S_n = L$ and if $\lim_{n\to\infty} t_n = M$ then $\lim_{n\to\infty} S_n t_n = LM$

proof: Given $S_n \to L$ and $t_n \to M$ as $n \to \infty$

$S_n + t_n \to L + M$ by Theorem 6.

$(S_n + t_n)^2 \longrightarrow (L+M)^2$ by Lemma 1.

Also $S_n - t_n \longrightarrow L - M$ by theorem 8

$(S_n - t_n)^2 \longrightarrow (L-M)^2$ by lemma

$\therefore (s_n+t_n)^2 - (s_n-t_n)^2 \to (L+M)^2 - (L-M)^2$

$= L^2+2LM+M^2-L^2+2LM-M^2$

$= 4LM.$

$\therefore s_n t_n = 1/4 \left[(s_n+t_n)^2-(s_n-t_n)^2\right] \to \frac{1}{4}(4ML)$

$= ML = LM$

as $n \to \infty$

Using the identity $ab = \frac{1}{4}\left[(a+b)^2-(a-b)^2\right]$

Lemma : (2)

If $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if $\lim_{n\to\infty} t_n = M$, given $M \neq 0$ then

$\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{M}$

Proof: Assume $M>0$, given $\epsilon>0$, we must find $N \in I \ni$:

$$\left|\frac{1}{t_n} - \frac{1}{M}\right| < \epsilon \quad (n \geq N)$$

(i.e) $\frac{|t_n-M|}{|t_n M|} < \epsilon \ (n \geq N) \to (1)$

since $\lim_{n\to\infty} t_n = M$, $\exists N_1 \in I \ni$:

$$|t_n - M| < M/2 \quad (n \geq N_1)$$

(i.e) $M - M/2 < t_n < M + \frac{M}{2}$

$\Rightarrow t_n > M/2 \quad (n \geq N_1)$

$\Rightarrow 1/t_n < 1/M/2 \quad (n \geq N_1)$

Again $\exists N_2 \in I \ni$:

$$|t_n - M| < \frac{M^2 \epsilon}{2} \quad (n \geq N_2)$$

Let $N = \max(N_1, N_2)$ we have for $n \geq N$

$$\frac{|t_n - M|}{|t_n M|} = \frac{1}{|t_n M|} |t_n - M|$$

$$< \frac{1}{M^2/2} \cdot \frac{M^2 \epsilon}{2} = \epsilon.$$

The case $M<0$ can be proved by applying the first case to $\{-t_n\}_{n=1}^{\infty}$

Hence the lemma proved.

theorem: 10

If $\{S_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ sequences of real numbers if $\lim_{n\to\infty} S_n = L$ and if $\lim_{n\to\infty} t_n = M$ where $M \neq 0$ then $\lim_{n\to\infty} \left(\frac{S_n}{t_n}\right) = \frac{L}{M}$

Proof: Since $\lim_{n\to\infty} t_n = M$ by the lemma (2)

$$\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{M} \quad (M \neq 0)$$

$\therefore$ By theorem (9), $\lim_{n\to\infty} \frac{S_n}{t_n} = \lim_{n\to\infty} S_n \cdot \frac{1}{t_n} = L \cdot \frac{1}{M}.$

theorem: 11:

(a) If $0<x<1$ then $\{x^n\}_{n=1}^{\infty}$ converges to 0

(b) If $0<x<\infty$, then $\{x_n\}_{n=1}^{\infty}$ diverges to ∞.

Proof: (a) If $0<x<1$ then $x^{n+1} = x \cdot x^n < x^n$

Hence $\{x^n\}_{n=1}^{\infty}$ is a non-increasing sequence which is bounded below.

By theorem (5) $\{x^n\}_{n=1}^{\infty}$ is convergent.

Let $L = \lim_{n\to\infty} x^n$.

By theorem (7). (taking $c = x$) it follows that

$\lim_{n\to\infty} x \cdot x^n = xL$.

(i.e) $\{x^{n+1}\}_{n=1}^{\infty}$ is a subsequence of $\{x^n\}_{n=1}^{\infty}$

$\therefore$ All subsequences of a convergent sequence of real numbers converges to the same limit we have $L = xL$ (i.e) $L(1-x) = 0$

$\because x \neq 1$, $L = 0$. this proves (a)

(b) If $x > 1$, then $x^{n+1} = x \cdot x^n > x^n$, so that $\{x^n\}_{}^{n}$ is non-decreasing.

We shall show that $\{x_n\}_{n=1}^{\infty}$ not bounded above.

For if $\{x_n^n\}_{n=1}^{\infty}$ were bounded above, then $\{x_n^n\}_{n=1}^{\infty}$ would converge to some limit $L \in R$

(By theorem 1)

But by the same reasoning as in (a) we can show that $L = Lx$ so that,

$L = 0 \lim_{n\to\infty} x^n$

But $x^n \geq 1$ so that $\{x_n\}_{n=1}^{\infty}$ clearly cannot converge to 0.

Thus $\Rightarrow\Leftarrow$ proves that $\{x^n\}_{n=1}^{\infty}$ is not bounded above.

$\therefore$ the sequence $\{x^n\}_{n=1}^{\infty}$ diverges to ∞ (by theorem 9)

2.8 Operations on Divergent sequences:

theorem : 12

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers that diverge to infinity then so do their sum and product.

(i.e) $\{s_n+t_n\}_{n=1}^{\infty}$ and $\{s_n \cdot t_n\}_{n=1}^{\infty}$ diverge to ∞.

Proof: Given $M>0$, choose $N_1 \in I \ni$:

$s_n > M \ (n \geq N_1)$ and choose $N_2 \in 2 \ni$:

$t_n \geq 1 \ (n \geq N_2)$

This is possible since both $s_n \to \infty$ and $t_n \to \infty$

Let $N = \max(N_1, N_2)$ we have.

$s_n \cdot t_n > M \cdot 1 > M \ (n \geq N)$

since M is an arbitary +ve number.

This proves the theorem.

theorem : 13

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\{s_n\}_{n=1}^{\infty}$ diverges to ∞, and if $\{t_n\}_{n=1}^{\infty}$ is bounded, then $\{s_n+t_n\}_{n=1}^{\infty}$ diverges to ∞.

Proof: Since $\{t_n\}_{n=1}^{\infty}$ $\exists$ $Q>0 \ni$:

$|t_n| \leq Q$, $(n \in J)$

i.e) $-Q < t_n < Q$ or

$$t_n > -Q \dashrightarrow (1)$$

Since $S_n \to \infty$,

Given $M > Q$, choose $(N \in I) \ni$;

$$S_n > M + Q \quad (n \geq N) \dashrightarrow (2)$$

Then for $n \geq N$, adding (1) & (2)

$S_n + t_n > M \quad (n \geq N)$

$\therefore S_n + t_n \to \infty$ as $n \to \infty$.

Corollary:

If $\{S_n\}_{n=1}^{\infty}$ diverges to ∞ and if $\{t_n\}_{n=1}^{\infty}$ converges then $\{S_n + t_n\}_{n=1}^{\infty}$ diverges to ∞.

proof: since $\{t_n\}_{n=1}^{\infty}$ converges it is bounded.

Hence from the above theorem B,

$\{S_n + t_n\}_{n=1}^{\infty}$ diverges to ∞

2.9 Limit Superior and Limit Inferior:

Defn: Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers that is bounded above and let

$M_n = \text{l.u.b}\{S_n, S_{n+1}, S_{n+2}, \dots\dots\}$

(a) If $\{M_n\}_{n=1}^{\infty}$ converges we define

$$\lim_{n \to \infty} \sup S_n = \lim_{n \to \infty} M_n$$

(b) If $\{M_n\}_{n=1}^{\infty}$ diverges to minus infinity we write $\limsup_{n\to\infty} S_n = -\infty$.

Note:

$\{M_n\}_{n=1}^{\infty}$ is non-increasing sequence ($M_n \geq M_{n+1}$) and hence either converges or diverges to $-\infty$

For example:

1. Let $S_n = (-1)^n$ $(n \in I)$. Then $\{S_n\}_{n=1}^{\infty}$ is bounded above by 1.

In this case $M_n = 1$, $\forall\, n \in I$ and hence

$\lim_{n\to\infty} M_n = 1$, $\quad \therefore \limsup_{n\to\infty} (-1)^n = 1$

2. consider the sequence $1, -1, 1, -2, 1, -3, 1, -4, \ldots\ldots$

Again $M_n = 1$ $\forall n$. So that limit superior of the sequence is 1.

3. If $S_n = -n$ $(n \in I)$ then,

$M_n = \text{l.u.b}\{-n, -n, -1, -n+2, \ldots\ldots\} = -n$

Hence $M_n \to -\infty$ as $n \to \infty$ and.

So $\limsup_{n\to\infty} (-n) = -\infty$

Defn: If $\{S_n\}_{n=1}^{\infty}$ is a sequence of real numbers that is not bounded above we write $\lim\sup S_n = \infty$. Clearly $\limsup_{n\to\infty} n = \infty$ $(\because M_n = \infty)$

Defn:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers that is bounded below and let

$m_n = g.l.b\{S_n, S_{n+1}, \ldots\}$

(a) If $\{m_n\}_{n=1}^{\infty}$ converges we define

$$\lim_{n\to\infty} \inf S_n = \lim_{n\to\infty} m_n$$

(b) If $\{m_n\}_{n=1}^{\infty}$ diverges to infinity we write

$$\lim_{n\to\infty} \inf S_n = \infty$$

Note: If $\{S_n\}_{n=1}^{\infty}$ is a sequence of real numbers that is not bounded below we write $\lim_{n\to\infty} \inf S_n = -\infty$

Examples:

(i) $\lim_{n\to\infty} \inf (-1)^n = -1$

(ii) $\lim_{n\to\infty} \inf n = \infty$

(iii) $\lim_{n\to\infty} \inf (-n) = -\infty$

(iv) The Sequence $1, -1, 1, -2, 1, -3, \ldots\ldots$ has $\lim \inf = -\infty$

Theorem : 14

If $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers then $\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} S_n$

Proof:

Let $\lim_{n\to\infty} S_n = L$

then given $\epsilon > 0$ $\exists N \in I \ni$:

$$|S_n - L| < \epsilon \quad (n \geq N)$$

i.e.) $L - \epsilon < S_n < L + \epsilon \quad (n \geq N)$

Thus if $n \geq N$ then $L+\epsilon$ is an upper bound for $\{S_n, S_{n+1}, S_{n+2}, \ldots\}$ and $L-\epsilon$ is not an upper bound.

Hence $L - \epsilon < M_n = \text{l.u.b}\{S_n, S_{n+1}, \ldots\} \leq L + \epsilon$

$$\therefore L - \epsilon < \lim_{n\to\infty} M_n \leq L + \epsilon$$

$$L - \epsilon < \lim_{n\to\infty} \sup S_n < L + \epsilon \quad \left[\because \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup S_n\right]$$

Since ϵ is arbitrary.

$$\lim_{n\to\infty} \sup S_n = \lim_{n\to\infty} S_n.$$

Theorem : 15

If $\{S_n\}_{n=1}^{\infty}$ is a convergent sequences of real numbers then $\lim_{n\to\infty} \inf S_n = \lim_{n\to\infty} S_n$

Proof: This proof is similar to the proof of theorem 14.

Note: From this theorem 14 & 15, it follows that if $\{S_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers then,

$$\limsup_{n\to\infty} S_n = \liminf_{n\to\infty} S_n = \lim_{n\to\infty} S_n = L$$

Theorem : 16

If $\{S_n\}_{n=1}^{\infty}$ is a sequences of real numbers then $\liminf_{n\to\infty} S_n \leq \limsup_{n\to\infty} S_n$

Proof:

If $\{S_n\}_{n=1}^{\infty}$ is bounded then.

$m_n = g.l.b\{S_n, S_{n+1}, S_{n+2}, \ldots\} \leq l.u.b\{S_{n+}\ldots \ldots\} = M_n$

i.e $m_n \leq M_n$ & $\lim_{n\to\infty} m_n \leq \lim_{n\to\infty} M_n$

Hence $\liminf_{n\to\infty} S_n \leq \limsup_{n\to\infty} S_n$

If $\{S_n\}_{n=1}^{\infty}$ is not bounded than either

$\liminf_{n\to\infty} S_n = \infty$ (or) $\liminf_{n\to\infty} S_n = -\infty$

Using the convention $-\infty < \infty$ we have

$$\liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n$$

Theorem : 17

If $\{S_n\}_{n=1}^{\infty}$ is a sequence of real numbers and if $\limsup_{n\to\infty} S_n = \liminf_{n\to\infty} S_n = L$, where $L \in R$ then $\{S_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} S_n = L$.

Proof: By hypothesis we have,

$$L = \limsup_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \text{l.u.b}\{S_n, S_{n+1}, S_{n+2}, \ldots\}$$

Thus given $\epsilon > 0$, $\exists N_1 \in I$ such that

$$|\text{l.u.b}\{S_n, S_{n+1}, S_{n+2}, \ldots\} - L| < \epsilon \quad (n \geq N_1)$$

$$S_n < L + \epsilon \quad (n \geq N_1) \ldots\ldots (1)$$

$||^{ly}$, since $\liminf_{n\to\infty} S_n = L$, $\exists N_2 \in I \ni$:

$$|\text{g.l.b}\{S_n, S_{n+1}, S_{n+2}, \ldots\} - L| < \epsilon \quad (n \geq N_2)$$

$$\Rightarrow S_n > L - \epsilon \quad (n \geq N_2) \ldots\ldots (2)$$

Let $N = \max(N_1, N_2)$

then from (1) & (2)

$$L - \epsilon < S_n < L + \epsilon \quad (n \geq N)$$

(i.e) $|S_n - L| < \epsilon \quad (n \geq N)$

This proves $\lim_{n\to\infty} S_n = L$

theorem : 18

If $\{S_n\}_{n=1}^{\infty}$ is a sequence of real no and $\limsup_{n\to\infty} S_n = \infty = \liminf_{n\to\infty} S_n$ then $\{S_n\}$ diverges to infinity.

Proof: since $\liminf_{n\to\infty} S_n = \infty$, given $M > 0$ $\exists N \in I$

$\ni$: $\text{g.l.b}\{S_n, S_{n+1}, S_{n+2}, \ldots\} > M \quad (n \geq N)$

$\Rightarrow$ that M is a lower bound but not g.l.b for $\{S_n, S_{n+1}, S_{n+2}, \ldots\}$ so that $S_n > m$ $(n \geq N)$ which proves the theorem.

theorem: 19

If $\{S_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are bounded Sequence of real numbers and if $S_n \leq t_n$ ($n \in$

then $\lim\limits_{n\to\infty} \sup S_n \leq \lim\limits_{n\to\infty} \sup t_n$ and $\lim\limits_{n\to\infty} \inf S_n \leq \lim\limits_{n\to\infty} \inf t_n$.

Proof: From the hypothesis $S_n \leq t_n$, it is clear that

$$\text{l.u.b}\{S_n, S_{n+1}, S_{n+2}, \ldots\} \leq \text{l.u.b}\{t_n, t_{n+1}, t_{n+2}, \ldots\} \quad \text{---}(1)$$

and

$$\text{g.l.b}\{S_n, S_{n+1}, S_{n+2}, \ldots\} \leq \text{g.l.b}\{t_n, t_{n+1}, t_{n+2}, \ldots\} \quad \text{---}(2)$$

Taking the limit as $n \to \infty$ is (1) & (2) we get

$$\lim_{n\to\infty} \sup S_n \leq \lim_{n\to\infty} \sup t_n \text{ and}$$

$$\lim_{n\to\infty} \inf S_n \leq \lim_{n\to\infty} \inf t_n$$

theorem: 20

If $\{S_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are bounded Sequences of real numbers then.

(i) $\lim\limits_{n\to\infty} \sup (S_n + t_n) \leq \lim\limits_{n\to\infty} \sup S_n + \lim\limits_{n\to\infty} \sup t_n$

(ii) $\lim\limits_{n\to\infty} \inf (S_n + t_n) \geq \lim\limits_{n\to\infty} \inf S_n + \lim\limits_{n\to\infty} \inf t_n$

Proof:

(i) Let $M_n = l.u.b \{s_n, s_{n+1}, s_{n+2}, \ldots\}$

$P_n = l.u.b \{t_n, t_{n+1}, t_{n+2}, \ldots\}$

then, $s_k \leq M_n, (k \geq n), t_k \leq P_n (k \geq n)$

So that,

$s_k + t_k \leq M_n + P_n \quad (k \geq n)$

$\Rightarrow$ that $M_n + P_n$ is an upper bound for

$\{s_n + t_n, s_{n+1} + t_{n+1}, s_{n+2} + t_{n+2}, \ldots\}$

So that,

$l.u.b \{s_n + t_n, s_{n+1} + t_{n+1}, \ldots\} \leq M_n + P_n$

$$\lim_{n\to\infty} l.u.b \{s_n + t_n, s_{n+1} + t_{n+1}, s_{n+2} + t_{n+2}, \ldots\} \leq \lim_{n\to\infty}(M_n + P_n)$$

$$= \lim_{n\to\infty} M_n + \lim_{n\to\infty} P_n$$

(or)

$$\lim_{n\to\infty} \sup (s_n + t_n) \leq \lim_{n\to\infty} \sup s_n + \lim_{n\to\infty} \sup t_n$$

III^{ly} we prove

$$\lim_{n\to\infty} \inf (s_n + t_n) \geq \lim_{n\to\infty} \inf s_n + \lim_{n\to\infty} \inf t_n$$

this theorem approach to define limit superior and limit inferior.

theorem: 21

Let $\{s_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers.

(1) If $\lim\limits_{n\to\infty} \sup S_n = M$ then for any $\epsilon > 0$,

(a) $S_n < M+\epsilon$ for all but a finite no of values of n.

(b) $S_n > M-\epsilon$ for infinitely many values of n

(2) If $\lim\limits_{n\to\infty} \inf S_n = m$ then for any $\epsilon > 0$

(c) $S_n > m-\epsilon$ for all but a finite no of values of n

(d) $S_n < m+\epsilon$ for infinitely many values of n

Proof: Let us prove part (2)

If (c) were false, then for some $\epsilon > 0$ we have $S_n \le m-\epsilon$ for infinitely many values of n.

So for any $n \in I$, the set $\{S_n, S_{n+1}, S_{n+2}, \dots\}$ would contain a term $\le m-\epsilon$

This implies that,

$g.l.b \{S_n, S_{n+1}, S_{n+2}, \dots\} \le m-\epsilon \quad (n \in I)$

Taking the limit as $n\to\infty$ we get

$\lim\limits_{n\to\infty} \inf S_n \le m-\epsilon$ which contradicts the hypothesis

$\therefore$ (c) is true

Suppose (d) is false.

then for some $\epsilon > 0$, $S_n < m+\epsilon$ for only a finite no of values of n.

So $\exists N \in I \ni$:

$$S_n \geq m + \epsilon \quad (n \geq N)$$

$$\therefore \lim_{n\to\infty} \inf S_n \geq m + \epsilon > m$$

which again contradicts hypothesis

thus (d) is true.

Note: The converse of the above theorem is true.

(i.e.) if $\{S_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers and if $M \in R$ is such that (a) and (b) hold for ~~an~~ every $\epsilon > 0$, then

$$\lim_{n\to\infty} \sup S_n = M$$

Similarly if (c) and (d) hold for every $\epsilon > 0$.

then $\lim_{n\to\infty} \inf S_n = m$

theorem: 22

Any bounded sequence of real numbers has a convergent subsequence.

Proof: Let $\{S_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and

Let $M = \lim_{n\to\infty} \sup S_n$

We shall construct a subsequence $\{S_{n_k}\}_{k=1}^{\infty}$ which converges to M

By thm 21. there are infinitely many values of n $\ni$: $s_n > M-1$

Let n_1 be one such value

(i.e) $n_1 \in I$ and $s_{n_1} > M-1$

$||^{ly}$ since there are infinitely many values of $n \ni$: $s_n > M-1/2$, we can find $n_2 \in I \ni$:

$n_2 > n_1$ and $s_{n_2} > M-1/2$

Proceeding in this way for each integer $k>1$ we can find $n_k \in I \ni$:

$n_k > n_{k-1}$ and

$s_{n_k} > M - 1/k$ ------> (1)

Gi $\epsilon > 0$ by (a) of theorem 21,

We can find $N \in I \ni$:

$s_n < M + \epsilon$ $(n \geq N)$ -----> (2)

choose $K \in I$ so that $1/K < \epsilon$ and $n_K > N$

then if $k \geq K$ we have $1/k < \epsilon$ and $n_k > N$

From (1) & (2) we get.

$M - \epsilon < M - 1/k < s_{n_k} < M + \epsilon$ $(k \geq K)$

(i.e) $|s_{n_k} - M| < \epsilon$ $(k \geq K)$

this proves $\lim_{k \to \infty} s_{n_k} = M$ which

Proves the theorem.

2.10. Cauchy Sequences:

Defn: Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{S_n\}_{n=1}^{\infty}$ is called a cauchy sequence if for any $\epsilon > 0$ $\exists$ an $N \in I \ni$:

$|S_m - S_n| < \epsilon \quad (m, n > N)$

Theorem: 2.3 1st

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ converges then $\{S_n\}_{n=1}^{\infty}$ is a cauchy sequence.

Proof: Let $L = \lim_{n \to \infty} S_n$

then given $\epsilon > 0$ $\exists$ an $N \in I \ni$:

$|S_m - L| < \epsilon/2 \quad (n \geq N)$

thus if $m, n \geq N$ we have

$|S_m - S_n| = |S_m - L + L - S_n|$

$\leq |S_m - L| + |L - S_n| < \epsilon/2 + \epsilon/2 = \epsilon$

(i.e.) $|S_m - S_n| < \epsilon \quad (m, n \geq N)$ which proves that $\{S_n\}_{n=1}^{\infty}$ is cauchy.

2nd 12th 6

Lamma:

If $\{S_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers then $\{S_n\}_{n=1}^{\infty}$ is bounded.

Proof: Given $\epsilon = 1$ choose $N \in I \ni$:

$|S_m - S_n| < 1 \quad (m, n \geq N)$

then $|S_m - S_N| < 1 \quad (m \geq N)$

Hence if $m \geq N$ we have

$$|S_m| = |(S_m - S_N) + S_N|$$
$$\leq |S_m - S_N| + |S_N|$$
$$\leq 1 + |S_N| \quad (m \geq N)$$

Let $M = \max\{|S_1|, \ldots\ldots |S_{N-1}|\}$ then

$|S_m| < M + 1 + |S_N| \quad (m \in I)$

$\therefore \{S_n\}_{n=1}^{\infty}$ is bounded

theorem: 24 2nd

If $\{S_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers, then $\{S_n\}_{n=1}^{\infty}$ is convergent

Proof:

By the above lemma, $\{S_n\}_{n=1}^{\infty}$ is bounded.

$\therefore \lim\limits_{n\to\infty} \sup S_n$ and $\lim\limits_{n\to\infty} \inf S_n$ are (finite) real number.

Hence to prove the theorem it is sufficient to show that $\lim\limits_{n\to\infty} \sup S_n = \lim\limits_{n\to\infty} \inf S_n$

By thm, w.k.t $\lim\limits_{n\to\infty} \sup S_n = \lim\limits_{n\to\infty} \inf S_n$

thus we need to prove that $\lim\limits_{n\to\infty} \sup S_n \leq \lim\limits_{n\to\infty} \inf S_n$ ------(1)

since $\{S_n\}_{n=1}^{\infty}$ is cauchy given

$\epsilon > 0 \; \exists N \in I \ni : |S_m - S_n| < \epsilon/2 \; (m, n \geq N)$

and so $|S_N - S_n| < \epsilon/2 \; (n \geq N)$

It follows that $S_N + \epsilon/2$ and $S_N - \epsilon/2$ are respectively upper and lower bounds for the set $\{S_N, S_{N+1}, S_{N+2}, \ldots\}$

$\Rightarrow$ for $n \geq N$,

$$S_N - \epsilon/2 \leq g.l.b\{S_N, S_{N+1}, S_{N+2}, \ldots\} \leq l.u.b\{S_N, S_{N+1}, S_{N+2}, \ldots\} \leq S_N + \epsilon/2$$

we must have

$$l.u.b\{S_N, S_{N+1}, S_{N+2}, \ldots\} - g.l.b\{S_N, S_{N+1}, S_{N+2}, \ldots\} \leq \epsilon$$

$$\therefore l.u.b\{S_N, S_{N+1}, S_{N+2}, \ldots\} \leq g.l.b\{S_N, S_{N+1}, S_{N+2}, \ldots\} + \epsilon$$

Taking lim as $n \to \infty$ on both side

$$\lim_{n\to\infty} \leq \lim_{n\to\infty} \inf S_n + \epsilon$$

Since ϵ is arbitary

$\therefore$ Hence the theorem.

UNIT
(X)

Nested Interval Theorem:

Theorem: 25

For each $n \in I$ let $I_n = [a_n, b_n]$ be a non empty closed bounded interval of real numbers such that

$I_1 \supset I_2 \supset I_3 \supset \dots \supset I_n \supset I_{n+1} \dots$

and $\lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty}$ (length of I_n) $= 0$

then $\bigcap_{n=1}^{\infty}$ contains precisely one point.

Proof: Given

$I \supset I_2 \supset I_3 \supset \dots \supset I_n \supset I_{n+1} \dots \longrightarrow (1)$

and $\lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty}$ (length of I_n) $= 0 \dots \to (2)$

By hypothesis (1) we have

$I_n \supset I_{n+1}$

$\therefore a_n \le a_{n+1} \le b_{n+1} \le b_n$

This shows that the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are respectively non-decreasing and non-increasing.

Moreover by (1) again all terms of both sequences lie in I_1 and hence the sequences are bounded.

(in unit I)

$\therefore$ By theorem (3) and (5) the sequences are convergent.

Let $x = \lim_{n\to\infty} a_n$ and $y = \lim_{n\to\infty} b_n$

then for any n, $a_n \le x$ and $y \le b_n$

By hypothesis (2) we have

$$y - x = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} (b_n - a_n) = 0$$

Thus $x = y$

Hence from (1) $a_n \le x$ and $x \le b_n$ $\forall n$

(i.e) $a_n \le x \le b_n$ for each n.

(i.e) $x \in [a_n, b_n] = I_n$ for each n which shows that $x \in \bigcap_{n=1}^{\infty} I_n$

Clearly no $z \ne x$ can lie in $\bigcap_{n=1}^{\infty} I_n$

Since by hypothesis (2) we have $|z - x|$ is greater than the length of I_n for sufficiently large n and hence z cannot lie in this I_n.

Hence $\bigcap_{n=1}^{\infty} I_n$ contains x and no other point.

This completes the proof of the theorem.

Example 1:

If a sequence $\{S_n\}_{n=1}^{\infty}$ diverges to ∞ and $S_n \neq 0$ for all $n \in N$, then the sequence $\{1/S_n\}_{n=1}^{\infty}$ converges to 0

Soln: Since $S_n \to \infty$, given $\epsilon > 0$, $\exists$ a positive integer $N \ni$:

$S_n > 1/\epsilon$ for $n \geq N$

(i.e) $1/S_n < \epsilon$ for $n \geq N$

(i.e) $\left|\frac{1}{S_n} - 0\right| < \epsilon$ for $n \geq N$

$\therefore$ the sequence $\{1/S_n\}_{n=1}^{\infty}$ converges to 0

Example: 2

If a sequence $\{S_n\}_{n=1}^{\infty}$ converges to 0 and $S_n > 0$ for $n \in N$, then the sequence $\{1/S_n\}_{n=1}^{\infty}$ diverges to ∞.

Proof: Since $S_n \to 0$.

Given $\epsilon > 0$ $\exists$ a +ve Integer $N \ni$:

$|S_n - 0| < \epsilon$ for $n \geq N$ ------(1)

But $S_n > 0$ $\forall n$ and $\therefore$ (1) becomes

$S_n < \epsilon$ for $n \geq N$

$1/S_n > 1/\epsilon$ for $n \geq N$

$\therefore \{1/S_n\}_{n=1}^{\infty}$ diverges to ∞

Example 3 to 18 see in book page No: 5.40

(S.G. Venkatchalapathy)

UNIT- III

SERIES OF REAL NUMBERS

k.rk

3.1 Convergence and Divergence:

Definition:

The infinite series $\sum_{n=1}^{\infty} a_n$ is an ordered pair $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $S_n = a_1 + a_2 + \dots + a_n$ $(n \in I)$

Here a_n is called the n^{th} term of the series and S_n is called the n^{th} partial sum of the series.

Definition:

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers with partial sums $S_n = a_1 + a_2 + \dots + a_n$ $(n \in I)$

If the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $A \in R$, we say that the series $\sum_{n=1}^{\infty} a_n$ converges to A. If $\{S_n\}_{n=1}^{\infty}$ diverges then, we say that $\sum_{n=1}^{\infty} a_n$ diverges.

Example: 1

Consider the series $1 + 1/2 + 1/2^2 + 1/2^3 + \dots$

This is a geometric series $a = a_1 = 1$ and common ratio $= 1/2 < 1$.

Now $S_n = 1 + 1/2 + 1/2^2 + \dots + 1/2^{n-1}$

$$= a\left(\frac{1-r^n}{1-r}\right)$$

$$= \frac{1 - 1/2^n}{1 - 1/2} \Rightarrow 2\left(1 - 1/2^n\right)$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1 - 1/2^n\right)$$

$$= 2(1-0) = 2$$

Example : 2

Consider the series $1+1+1+\ldots\ldots$

$$S_n = \underbrace{1+1+1+\ldots\ldots}_{n \text{ times}} = n$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$$

$\therefore$ lim the given series diverges to ∞.

Theorem : 2

If $\sum_{n=1}^{\infty} a_n$ is convergent series, then

$$\lim_{n\to\infty} a_n = 0$$

Proof:

Suppose $\sum_{n=1}^{\infty} a_n = A$ then $\lim_{n\to\infty} S_n = A$

(ie) $\lim_{n\to\infty} a_1 + a_2 + \ldots\ldots + a_n = A$.

$$a_n = (a_1 + a_2 + \ldots\ldots + a_n) - (a_1 + a_2 + \ldots\ldots + a_{n-1})$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (a_1 + a_2 + \ldots\ldots + a_n) - \lim_{n\to\infty} (a_1 + \ldots\ldots + a_{n-1})$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= A - A$$

$$= 0$$

Remark:

1. From the above theorem, it follows that if a series $\sum_{n=1}^{\infty} a_n$ is s; $\lim_{n\to\infty} a_n \neq 0$ then the series is not convergent.

For eg: Let $a_n = \frac{n-1}{n+1}$ then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{1-1/n}{1+1/n}$$

$$= 1 \neq 0$$

$\therefore$ The series is not convergent.

2. Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = (-1)^n$

Here $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n$ does not exist.

$\therefore$ The series cannot converge.

3. It is to be remembered that the condition $\lim_{n\to\infty} a_n = 0$ is only necessary for convergence but not a sufficient condition.

For example: the series $\sum_{n=1}^{\infty} 1/n$ is not convergent even though $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

3.2 Series with non-negative terms:

Theorem 3: If $\sum_{n=1}^{\infty} a_n$ is a series of non-negative numbers with $S_n = a_1 + a_2 + \dots + a_n$ $(n \in I)$ then.

(a) $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{S_n\}_{n=1}^{\infty}$ is

(b) $\sum_{n=1}^{\infty} a_n$ diverges if $\{S_n\}_{n=1}^{\infty}$ is not bounded.

Proof: (a) Since $a_{n+1} \geq 0$ we have $S_{n+1} = a_1 + a_2 + \dots + a_n + a_{n+1} = S_n + a_{n+1} \geq S_n$

Thus $\{S_n\}_{n=1}^{\infty}$ is non-decreasing sequence and by hypothesis bounded.

W.K.T, every non-decreasing sequence bounded above is convergent.

$\therefore \{S_n\}_{n=1}^{\infty}$ is convergent and hence $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\{S_n\}_{n=1}^{\infty}$ is not bounded than $\{S_n\}_{n=1}^{\infty}$ diverges.

($\because$ every unbounded increasing sequence diverges)

Hence so does $\sum_{n=1}^{\infty} a_n$ diverges.

Remarks:-

The above theorem says that a series of positive terms is either convergent or divergent.

$\therefore$ if the series is not convergent then it must be divergent.

For example,

consider the series $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$

Here $a_n = \sqrt{\frac{n}{n+1}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}}$$

$$= 1 \neq 0$$

$\therefore$ the series is not convergent.

But the given series is a series of +ve term.

$\therefore$ It diverges.

Theorem : 4

(a) If $0 < x < 1$, then $\sum_{n=1}^{\infty} x^n$ converges to $1/1-x$

(b) If $x \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges.

Proof: To prove (a) we have.

$S_n = 1 + x + \dots + x^n$

$\therefore S_n = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$

But if $0<x<1$ then $\lim_{n\to\infty} x^{n+1} = 0$.

(By theorem namely (a) if $0<x<1$ then $\{x^n\}_{n=1}^{\infty}$ converges to 0 (b) if $1<x<\infty$ then $\{x^n\}_{n=1}^{\infty}$ diverges to infinity).

Hence $\lim_{n\to\infty} S_n = \frac{1}{1-x}$ which proves (a)

(b) since $x \geq 1$ then $\{x^n\}_{n=1}^{\infty}$ does not converge to 0.

(ie) $\lim_{n\to\infty} x^n \neq 0$.

Hence S_n diverges.

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theorem: 5

the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

proof:

In this case $S_n = 1 + 1/2 + 1/3 + \cdots + 1/n$.

Let us examine the subsequence.

$S_1, S_2, S_4, S_8, \ldots \ldots S_n \ldots \ldots$

we have. $S_1 = 1$

$S_2 = 1 + 1/2 = 3/2$

$S_4 = S_2 + 1/3 + 1/4 > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2$

$S_8 = S_4 + 1/5 + 1/6 + 1/7 + 1/8 > 2 + 1/8 + 1/8 + 1/8 + 1/8 = 5/2$

In general it can be shown by induction that $S_{2^n} > \frac{n+2}{2}$

Also $S_{2^n} \to \infty$ as $n \to \infty$.

Thus $\{S_n\}_{n=1}^{\infty}$ contains a divergent sequence and hence diverges.

Note: the series $1+1/2+1/3+\ldots\ldots+1/n+\ldots\ldots$ is known as the harmonic.

Notation:

For series with non-negative terms only the following notation is being introduced.

If $\sum_{n=1}^{\infty} a_n$ is an convergent series of non-negative numbers we write $\sum_{n=1}^{\infty} a_n < \infty$.

If $\sum_{n=1}^{\infty} a_n$ is a divergent series of non negative numbers we write $\sum_{n=1}^{\infty} a_n = \infty$

thus $\sum_{n=0}^{\infty} (1/2)^n < \infty$, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

Theorem: 6

If $\sum_{n=1}^{\infty} a_n$ is a divergence series of positive numbers then there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of a positive number which converges to zero but for which $\sum_{n=1}^{\infty} \varepsilon_n a_n$ still diverge.

Proof:

Let $S_n = a_1 + a_2 + \ldots\ldots + a_n$.

Let us show that the series $\sum_{k=1}^{\infty} \frac{S_{k+1} - S_k}{S_{k+1}}$ diverges

For any $m \in I$ let us choose $n \in I \ni$:
$S_{n+1} > 2S_m$. This is possible since by $\{S_k\}_{k=1}^{\infty}$ diverges to infinity and $\{S_k\}_{k=1}^{\infty}$ is non-decreasing

$$\therefore \sum_{k=m}^{n} \frac{S_{k+1}-S_k}{S_{k+1}} \geq \sum_{k=m}^{n} \frac{S_{k+1}-S_k}{S_{n+1}}$$

$$= \frac{1}{S_{n+1}}\Big[(S_{m+1}-S_m)+(S_{m+2}-S_{m+1})+\cdots + (S_{n+1}-S_n)\Big]$$

$$= \frac{S_{n+1}-S_m}{S_{n+1}} \geq \frac{S_{n+1}-\frac{1}{2}S_{n+1}}{S_{n+1}} \geq 1/2$$

Thus for any $m \in I$, $\exists\, n \in I \ni$:

$$\sum_{k=m}^{n} \frac{S_{k+1}-S_k}{S_{k+1}} \geq \frac{1}{2}$$

This implies that the partial sum of the series do not form a cauchy sequence and hence

$$\sum_{k=1}^{\infty} \frac{S_{k+1}-S_k}{S_{k+1}} = \infty.$$

But $S_{k+1}-S_k = (a_1+\cdots+a_k+a_{k+1}) - (a_1+\cdots+a_k)$

$= a_{k+1}$

$$\text{thus } \sum_{k=1}^{\infty} \frac{a_{k+1}}{S_{k+1}} = \sum_{k=2}^{\infty} \frac{a_k}{S_k} = \infty$$

Let $\epsilon_k = 1/S_k$ then $\epsilon_k \to 0$ as $k \to \infty$

and $\sum_{k=2}^{\infty} \epsilon_k \cdot a_k = \infty$

$\therefore$ Hence proved.

6th

3.3 Alternating Series:

Defn:

An alternating series is an infinite series whose terms alternate in sign.

For example the series.

(i) $1 - 1/2 + 1/3 - 1/4 + \cdots\cdots$

(ii) $1 - 2 + 3 - 4 + \cdots\cdots$

(iii) $1 - 1/2 + 1/4 - 1/8 + \cdots\cdots$ are all alternating series.

An alternating series may be written as

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$

[or as $\sum_{n=1}^{\infty} (-1)^n a_n$ if the first term is negative] where each a_n is positive.

VVI

Theorem 7:

Leibnitz Theorem for Alternating Series:

Statement:

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that.

(a) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots\cdots$

(i.e) $\{a_n\}_{n=1}^{\infty}$ is non increasing and

(b) $\lim_{n\to\infty} a_n = 0$ then the alternating series

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

proof:-

Let us consider first the partial sums with odd index $S_1, S_3, S_5, \ldots\ldots$ we have

$S_3 = S_1 - a_2 + a_3$

$\because a_3 \leq a_2$ by hypothesis (a)

$S_1 \geq S_3$

we get $S_3 \leq S_1$

In general for any $n \in I$ we have

$S_{2n+1} = S_{2n-1} - a_{2n} + a_{2n+1} \leq S_{2n-1}$

Thus $S_1 \geq S_3 \geq \ldots \geq S_{2n-1} \geq S_{2n+1} \ldots\ldots$

So that $\{S_{2n-1}\}_{n=1}^{\infty}$ is non-increasing.

Also $S_{2n-1} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2n-3} - a_{2n-2}) + a_{2n-1}$

since each quantity within the bracket is non-negative and $a_{2n+1} > 0$.

$\therefore S_{2n-1} > 0$

Hence $\{S_{2n-1}\}_{n=1}^{\infty}$ is convergent.

[$\because$ a non-increasing sequence which is bounded below is convergent]

III^{ly} the sequence $S_2, S_4, \ldots\ldots S_{2n}, \ldots\ldots$ is convergent.

For $S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2} \geq S_{2n}$

so $\{S_{2n}\}_{n=1}^{\infty}$ is non-decreasing.

Also $S_{2n} = a_1 - (a_2 - a_3) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$

Each quantity inside the bracket is non-negative.

then $S_{2n} \le a_1$. So that $\{S_{2n}\}_{n=1}^{\infty}$ is bounded above.

Now, Let $\lim_{n\to\infty} S_{2n-1} = M$ and $\lim_{n\to\infty} S_{2n} = L$

By hypothesis (b)

$$0 = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} S_{2n-1} = L - M$$

(i.e) for $L = M$ and so both $\{S_{2n}\}_{n=1}^{\infty}$ and $\{S_{2n-1}\}_{n=1}^{\infty}$ converges to L

[i.e) $\epsilon > 0$, we can find $N_1, N_2 \in I \ni$:

$|S_{2n} - L| < \epsilon$ for $2n > N_1$ and

$|S_{2n-1} - L| < \epsilon$ for $2n-1 > N_2$

combining these two and taking $N = \max(N_1, N_2)$ we get $|S_n - L| < \epsilon$ for $n > N$]

this that $\{S_n\}_{n=1}^{\infty}$ converges to L and hence $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

corollary:

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ satisfies the hypothesis of the above theorem and hence converges to some $L \in R$ then $|S_k - L| \le a_{k+1}$ $(k \in I)$

Proof:

S_{2n-1} is a decreasing sequence converging to g.l.b and S_{2n} is an increasing sequence converging to l.u.b

$\therefore S_{2n-1} \ge L$ and $S_{2n} \le L$.

Thus $0 \le S_{2n-1} - L \le S_{2n-1} - S_{2n} = a_{2n}$

$\therefore |S_{2n-1} - L| \le a_{2n}$

|||ly $0 \le L - S_{2n} \le S_{2n+1} - S_{2n} = a_{2n+1}$

so that $|S_{2n} - L| \le a_{2n+1}$

thus whether k is odd or even.

we have shown that $|S_k - L| \le a_{k+1}$ $(k \in I)$

3.4 conditional convergence and Absolute convergence.

Defn:

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers.

(a) If $\sum_{n=1}^{\infty} |a_n|$ converges we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b): If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges

we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally

Example:

(i) consider the series

$1 - 1/2 + 1/4 - 1/8 + \cdots\cdots$ converges ------> (1)

If we take absolute value of each term, the series

$1 + 1/2 + 1/4 + 1/8 + \cdots\cdots$ converges ------> (2)

∴ we say that the series (1) converges absolutely.

(ii) consider the series

$1 - 1/2 + 1/3 - 1/4 + \cdots\cdots$ ------> (3)

which converges while the series

$1 + 1/2 + 1/3 + 1/4 + \cdots\cdots$ ------> (4)

∴ we say that the series (3) diverges converges conditionally.

theorem: 8

If $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} a_n$ converges

proof: Let $S_n = a_1 + a_2 + \cdots\cdots + a_n$

we have to show that $\{S_n\}_{n=1}^{\infty}$ converges

For this it is enough we show that $\{S_n\}_{n=1}^{\infty}$ is cauchy.

Given $\sum_{n=1}^{\infty} |a_n| < \infty$.

This implies that $\{t_n\}_{n=1}^{\infty}$ converges

where $t_n = |a_1| + |a_2| + \dots + |a_n|$

$\therefore \{t_n\}_{n=1}^{\infty}$ is cauchy.

Thus given $\epsilon > 0$ $\exists$ $N \in I \ni$:

$|t_m - t_n| < \epsilon$ $(m, n \geq N)$

If $m > n$, $|S_m - S_n| = |a_{m+1} + a_{m+2} + \dots + a_m|$

$\leq |a_{n+1}| + |a_{n+2}| + \dots + |a_n|$

$= |t_m - t_n| < \epsilon$

$\therefore |S_m - S_n| < \epsilon$ $(m, n \geq N)$

Hence $\{S_n\}_{n=1}^{\infty}$ is cauchy, which completes the proof.

Positive and Negative components of a series:

Let us separate $\sum_{n=1}^{\infty} a_n$ into the series of positive a_n and the series of negative a_n as follows:

If $\sum_{n=1}^{\infty} a_n$ is a series of real numbers.

Let $P_n = a_n$ if $a_n > 0$

$P_n = 0$ if $a_n \leq 0$

Similarly:

let $q_n = a_n$ if $a_n \leq 0$

$q_n = 0$ if $a_n > 0$

thus for the series $1 - 1/2 + 1/3 - 1/4 + \cdots\cdots$

$p_1 = 1,\ p_3 = 1/3, \cdots\cdots p_{2n-1} = \frac{1}{2n-1}$

$p_2 = p_4 = \cdots\cdots = 0$ and $q_2 = -1/2,\ q_4 = -1/4 \cdots q_{2n} = \frac{1}{2}$

thus p_n are the positive terms of $\sum_{n=1}^{\infty} a_n$,

while q_n are the negative terms.

It is easy to see that

$p_n = \max(a_n, 0) \quad q_n = \min(a_n, 0)$

(or) $2p_n = a_n + |a_n|$ and $2q_n = a_n - |a_n|$

Theorem: 9

(a) If $\sum_{n=1}^{\infty} a_n$ converges absolutely then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge.

(b) If $\sum_{n=1}^{\infty} a_n$ converges conditionally then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverge.

Proof:

(a) Since $\sum_{n=1}^{\infty} a_n$ converges absolutely, both by theorem 8 $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge.

$\therefore$ By thm (1), $\sum_{n=1}^{\infty} (a_n - |a_n|)$ converge implying that $\sum_{n=1}^{\infty} q_n$ converges.

This proves (a)

(b) By hypothesis $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Now $2p_n = a_n + |a_n|$

(or) $|a_n| = 2p_n - a_n$

If we assume that $\sum_{n=1}^{\infty} p_n$ converges then by theorem (1)

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} (2p_n - a_n)$$

$$= 2\sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} a_n \text{ would converge,}$$

which contradicts our assumption.

Hence $\sum_{n=1}^{\infty} p_n$ diverges.

3.6 Test for Absolute convergence.

Defn:

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of real numbers we say that $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ [or that $\sum_{n=1}^{\infty} b_n$ dominated by $\sum_{n=1}^{\infty} a_n$]

$\exists\ N \in I \ni : |a_n| \le |b_n| \ (n \ge N)$

In this case we write.

$$\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$$

Example:-

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \ll \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

Since $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2} \le \frac{1}{2n+1}$ for $n > 3$.

Theorem : 10 Comparison Test

Statement:-

If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ and if $\sum_{n=1}^{\infty} b_n$ converges absolutely the $\sum_{n=1}^{\infty} a_n$ also converges absolutely symbolically:

if $\sum_{n=1}^{\infty} a_n << \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |b_n| < \infty$ then $\sum_{n=1}^{\infty} |a_n| < \infty$

Proof:-

Since $\sum_{n=1}^{\infty} b_n$ converges absolutely

let $M = \sum_{n=1}^{\infty} |b_n|$

Given that $|a_n| \leq |b_n|$ for $n \geq N$.

Hence if $S_n = |a_1| + \dots\dots + |a_n|$, for $n \geq N$.

then, $S_n = |a_1| + |a_2| + \dots\dots + |a_N| + |a_{N+1}| + \dots\dots + |a_n|$

$S_n \leq |a_1| + \dots\dots + |a_N| + |b_{N+1}| + \dots\dots + |b_N|$

$\leq |a_1| + \dots\dots\dots + |a_N| + M$.

This means that the sequence of partial sums of $\sum_{n=1}^{\infty} |a_n|$ is bounded above and hence converges.

i.e.) $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Theorem : 11

If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |a_n| = \infty$ then $\sum_{n=1}^{\infty} |b_n| = \infty$

proof:

If $\sum_{n=1}^{\infty} |b_n|$ converges, then by the above comparison test,

$\sum_{n=1}^{\infty} |a_n|$ also converges

This contradiction shows that

$\sum_{n=1}^{\infty} |b_n|$ diverges

theorem: 12 **Comparison test - limit form**

Statement:-

(a) If $\sum_{n=1}^{\infty} b_n$ converges absolutely and if $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists, then $\sum_{n=1}^{\infty} a_n$ converges absolutely

(b) If $\sum_{n=1}^{\infty} |a_n| = \infty$ and if $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists then $\sum_{n=1}^{\infty} |b_n| = \infty$

proof:-

(a) Since $\lim_{n\to\infty} \frac{|a_n|}{|b_n|}$ exists $\left\{ \frac{|a_n|}{|b_n|} \right\}_{n=1}^{\infty}$ is bounded.

$\therefore \exists M > 0 \ni \left| \frac{a_n}{b_n} \right| \leq M$

i.e) $|a_n| \leq M |b_n|$ $(n \in I)$

this means that $\sum_{n=1}^{\infty} a_n$ is dominated by the absolutely convergent series $\sum_{n=1}^{\infty} M b_n$.

By theorem 10, $\sum |a_n| < \infty$

(b) As in part (a) we have $|a_n| \leq m|b_n|$

So that $\sum_{n=1}^{\infty} |b_n|$ dominates $\sum_{n=1}^{\infty}\left(\frac{1}{m}\right)|a_n|$ which

diverges. Hence $\sum |b_n| = \infty$

Theorem: 13 Ratio Test

Statement:

Let $\sum_{n=1}^{\infty} a_n$ be a series of non zero real

numbers and let $a = \lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right|$

$A = \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$. So that $a \leq A$ then

(a) If $A < 1$ then $\sum_{n=1}^{\infty} |a_n| < \infty$

(b) If $a > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

(c) If $a \leq 1 \leq A$, the test fails.

(i.e) nothing can be said about the convergence

Proof:

Let $A < 1$ choose any $B \ni: A < B < 1$

then $B = A + \varepsilon$ for some $\varepsilon > 0$.

$\therefore$ By a known result on limit superior

theorem 21 (unit II)

$\exists\, N \in I \ni: \left|\frac{a_{n+1}}{a_n}\right| \leq B \quad (n \geq N)$

then $\left|\frac{a_{N+1}}{a_N}\right| \leq B, \left|\frac{a_{N+2}}{a_{N+1}}\right| \leq B$

and $\left|\frac{a_{N+2}}{a_N}\right| = \left|\frac{a_{N+2}}{a_{N+1}}\right| \cdot \left|\frac{a_{N+1}}{a_N}\right| \leq B^2$

for any $k \geq 0$ we have

$$\left|\frac{a_{N+k}}{a_N}\right| = \left|\frac{a_{N+k}}{a_{N+k-1}}\right| \cdots\cdots \left|\frac{a_{N+1}}{a_N}\right| \leq B^k$$

thus $|a_{N+k}| \leq |a_N| B^k \quad (k=0,1,2\ldots\ldots)$

But $\sum_{k=0}^{\infty} |a_n| \cdot B^k$ converges. since $0<B<1$

and $\sum_{k=0}^{\infty} B^k$ is a geometric series

thus $\sum_{k=0}^{\infty} |a_{N+k}|$ converges

(i.e) $|a_N| + |a_{N+1}| + |a_{N+2}| + \ldots\ldots\ldots$ converges

$\therefore |a_1| + |a_2| + \ldots\ldots + |a_N| + |a_{N+1}| + \ldots\ldots$ converges.

It follows easily that $\sum_{n=1}^{\infty} |a_n| < \infty$

this proves (a)

(b) If $\alpha > 1$ then by a result on limit inferior

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \forall\, n \geq N.$$

then $|a_N| < |a_{N+1}| < |a_{N+2}| < \ldots\ldots$

Since $a_N \neq 0$, $\lim_{n\to\infty} a_N \neq 0$.

thus by theorem (4), $\sum_{n=1}^{\infty} a_n$ diverges.

(c) To illustrate part (c)

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1/n$.

Here we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/n+1}{1/n} = 1$

$\alpha = 1 = A$

The Series diverges

Let us consider the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = 1$$

$\therefore \alpha = 1 = A$

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

Remark:

From the above theorem we find that if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and equal to L then $\sum_{n=1}^{\infty} |a_n|$ converges if $L<1$ and diverges if $L>1$.

If $L=1$, nothing can be said.

Theorem: 14 Root test

Statement:

If $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = A$ then the series of real numbers $\sum_{n=1}^{\infty} a_n$

(a) converges absolutely if $A<1$.

(b) diverges if $A>1$.

This includes the case $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = A$

If $A=1$, test fails

Proof: let $A<1$. Choose B so that $A<B<1$.

then by a result regarding limit superior

$\exists N \in I \ni: \sqrt[n]{|a_n|} < B \ (n \geq N)$

$\Rightarrow |a_n| < B^n \quad (n \geq N)$

thus $\sum_{n=1}^{\infty} |a_n|$ is dominated by $\sum_{n=1}^{\infty} B^n$ which

converges (absolutely)

By theorem (10) $\sum_{n=1}^{\infty} |a_n| < \infty$ which proves (a)

For (b) if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$.

Again by a known result on limit superior

$\sqrt[n]{|a_n|} > 1$ for infinitely many values of n.

$\Rightarrow |a_n| > 1$ for infinitely many n.

$\therefore \{a_n\}_{n=1}^{\infty}$ does not converge to 0.

$\therefore$ By a known theorem $\sum_{n=1}^{\infty} a_n$ diverges.

this proves (b)

Theorem: 15

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then (a) if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$, the series $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for all real x;

(b) if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L > 0$ then $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for $|x| < 1/L$ and diverges for $|x| > \frac{1}{L}$

(c) If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then $\sum_{n=1}^{\infty} a_n x^n$ converges only for $x = 0$ and diverges for all other x.

Proof: To prove (a)

We have $\sqrt[n]{|a_n x^n|} = |x| \sqrt[n]{|a_n|}$

$\therefore$ if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$

then for any x, $\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x| = 0 = 0 < 1$

$\therefore$ By the previous theorem, the series

$\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for all real x

this proves (a)

To prove (b)

$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = |Lx|$

By the above theorem,

$\sum_{n=1}^{\infty} a_n x^n$ will converge absolutely

if $|Lx| < 1$ i.e. $|x| < 1/L$ and diverges

if $|Lx| >$ i.e. $|x| > 1/L$ This proves (b)

To prove (c), we note that for $x=0$, the series

converges to a_0

If $x \neq 0$, then let $M = \frac{1}{|x|}$

Since $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ we can select N

$\ni$: for $n \geq N$

$\sqrt[n]{|a_n|} > M = \frac{1}{|x|} \Rightarrow |a_n x^n| > 1$

This shows that the n^{th} term of the series

$\sum a_n x^n$ does not tend to zero.

Hence the series diverges.

corollarly:

If the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = x_0$ then it converges absolutely for all $x \ni |x| < |x_0|$

theorem : 1

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to A and B respectively then (i) $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $A+B$

(ii) $\sum_{n=1}^{\infty} (a_n - b_n)$ converges to $A-B$

proof:-

If $S_n = a_1 + a_2 + \cdots\cdots + a_n$ and

$t_n = b_1 + b_2 + \cdots\cdots + b_n$ then by hypothesis of the theorem.

$\lim_{n\to\infty} S_n = A$, $\lim_{n\to\infty} t_n = B$

the n^{th} partial sum of $\sum_{n=1}^{\infty} (a_n + b_n)$ is

$$(a_1+b_1)+(a_2+b_2)+\cdots\cdots+(a_n+b_n) = (a_1+a_2+\cdots\cdots+a_n) + (b_1+b_2+\cdots\cdots+b_n)$$

$$= S_n + t_n$$

By a known theorem on sequences

$$\lim_{n\to\infty} (S_n + t_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} t_n$$

$$= A + B$$

this proves $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

m^{ly} (ii) proved.

14/09/2020
Monday

UNIT - IV

SERIES OF REAL NUMBERS (contd.)

3.7 Series whose terms form a non-increasing Sequence.

Theorem: 1

If $\{a_n\}_{n=1}^{\infty}$ is a non increasing sequence of positive numbers and if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

proof: We have $a_1 \geq a_2 \geq a_3 \geq \ldots\ldots$

$\therefore a_1 \leq a_1$

$a_2 + a_3 \leq a_2 + a_2 = 2a_2$

$a_4 + a_5 + a_6 + a_7 \leq a_4 + a_4 + a_4 + a_4 = 4a_4 = 2^2 a_{2^2}$

for any $n \in I$, $a_{2^n} + a_{2^{n+1}} + \ldots + a_{2^{n+1}} \leq 2^n a_2$

These inequalities imply that

$$\sum_{k=1}^{2^{n+1}-1} a_k \leq \sum_{k=0}^{n} 2^k a_{2^k} \leq \sum_{k=0}^{\infty} 2^k a_{2^k}$$

Hence for any $m \in I$

$$\sum_{k=1}^{m} a_k \leq \sum_{k=0}^{n} 2^k a_{2^k}$$

But by hypothesis $\sum_{k=0}^{\infty} 2^k a_{2^k} < \infty$

$\therefore$ By comparison test

$\sum_{k=1}^{\infty} a_k$ converges.

the converse of the above theorem is also true.

Theorem: 2

If $\{a_n\}_{n=1}^{\infty}$ is non increasing sequence of positive ~~integer~~ numbers and if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof:

We have $a_3 + a_4 \geq 2a_4$

$$a_5 + a_6 + a_7 + a_8 \geq 4a_8$$

in general,

$$a_{2^n+1} + \cdots\cdots + a_{2^{n+1}} \geq 2^n a_{2^{n+1}} = \frac{1}{2}\left(2^{n+1} a_{2^{n+1}}\right)$$

So that $\sum_{k=3}^{2^{n+1}} a_k \geq \frac{1}{2} \sum_{k=1}^{n} 2^{k+1} a_{2^{k+1}} = \frac{1}{2} \sum_{k=2}^{n+1} 2^k a_{2^k}$

$\therefore \sum_{k=1}^{\infty} a_k \geq 1/2 \sum_{k=0}^{\infty} 2^k a_{2^k}$

But $\sum_{k=0}^{\infty} 2^k a_{2^k}$ diverges

$\therefore$ By comparison test $\sum_{k=1}^{m} a_k$ diverges.

Corollary: The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Proof: Here $a_n = \frac{1}{n^2}$

$\therefore \sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n < \infty$

$\therefore$ By the above theorem,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Note 1:

consider $\sum_{n=1}^{\infty} \frac{1}{n}$ Here $a_n = \frac{1}{n}$

$$\therefore \sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} 1 = \infty.$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem : 5

If $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence of positive numbers and if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} na_n = 0$.

proof:

Let $S_n = a_1 + a_2 + \dots + a_n$

If $\sum_{n=1}^{\infty} a_n = A$ then

$$\lim_{n\to\infty} S_n = A = \lim_{n\to\infty} S_{2n} \quad \text{------->} (1)$$

Thus $\lim_{n\to\infty} (S_{2n} - S_n) = 0$

Now $S_{2n} - S_n = a_{n+1} + a_{n+2} + \dots + a_{2n}$

$\geq a_{2n} + a_{2n} + \dots + a_{2n}$

$\because$ (the Sequence $\{a_n\}_{n=1}^{\infty}$ is non-increasing)

$\therefore 0 \leq na_{2n} \leq S_{2n} - S_n$

Taking limit as $n \to \infty$ and using (1) we

This implies $\lim_{n\to\infty} n a_{2n} = 0$

and $\lim_{n\to\infty} 2n\, a_{2n} = 0$ --------> (2)

But $a_{2n+1} \le a_{2n}$.

$(2n+1) a_{n+2} \le \left(\frac{2n+1}{2n}\right) 2n a_{2n}$

(By 2)

$\therefore \lim_{n\to\infty} (2n+1)\, a_{2n} = 0$ --------> (3)

From (2) & (3) we get,

$\lim_{n\to\infty} n a_n = 0.$

Problems:

Examples Book page No: 3.26.

3.10 The class l^2

Defn: The class l^2 is the class of all sequences $s = \{s_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} s_n^2 < \infty$

thus the elements of l^2 are sequences.

Example: 1

The sequence 0, 0, 0, ... is clearly an element of l^2.

Example: 2

The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is an element of l^2

since $\sum_{n=1}^{\infty} 1/n^2 < \infty$.

Example :3:

clearly the sequence $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is not an element of l^2.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem 4:

the Schwartz inequality

Statement:

If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in l^2, then $\sum_{n=1}^{\infty} s_n t_n$ is absolutely convergent and $\left|\sum_{n=1}^{\infty} s_n t_n\right| \leq \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}$ ------→(1)

Proof:

Let us assume that atleast one s_n say $s_N \neq 0$ otherwise the theorem is trivial.

For fixed $n \geq N$ and any $x \in R$ we have $\sum_{k=1}^{n} (x s_k + t_k)^2 \geq 0$

(i.e) $x^2 \sum_{k=1}^{n} s_k^2 + 2x \sum_{k=1}^{n} s_k t_k + \sum_{k=1}^{n} t_k^2 \geq 0$

this is of the form $Ax^2 + Bx + C \geq 0$ where

$A = \sum_{k=1}^{n} s_k^2 > 0$, $B = 2\sum_{k=1}^{n} s_k t_k$, $C = \sum_{k=1}^{n} t_k^2$

[From calculus w.k.T the minimum value of $Ax^2 + Bx + C$ $(A > 0)$ occurs when $x = -\frac{B}{2A}$]

Setting $x = -\frac{B}{2A}$ we get

$$A\left(\frac{-B}{2A}\right)^2 + B\left(\frac{-B}{2A}\right) + C \geq 0$$

(or)

$$B^2 \leq 4AC$$

(i.e) $$\left(\sum_{k=1}^{n} s_k t_k\right)^2 \leq \left(\sum_{k=1}^{n} s_k^2\right)\cdot\left(\sum_{k=1}^{n} t_k^2\right) \dashrightarrow (2)$$

Replacing $s_k \cdot t_k$ by $|s_k|\,|t_k|$ in (2)

we obtain

$$\sum_{k=1}^{n} |s_k t_k| \leq \left(\sum_{k=1}^{n} s_k^2\right)^{1/2} \left(\sum_{k=1}^{n} t_k^2\right)^{1/2}$$

$$\leq \left(\sum_{k=1}^{\infty} s_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} t_k^2\right)^{1/2}$$

Thus the sequence of partial sums of $\sum_{k=1}^{\infty} |s_k t_k|$ is bounded and hence $\sum_{k=1}^{\infty} |s_k t_k| < \infty$

(i.e) $\sum_{k=1}^{\infty} s_k t_k$ converges.

Letting n to approach infinity in (2)

we obtain (1)

$$\left|\sum_{n=1}^{\infty} s_n t_n\right| \leq \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} \left(\sum_{k=1}^{n} t_k^2\right)^{1/2}$$

Hence proved.

(19)

Theorem: 5 The Minkowski Inequality

Statement:

If $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in l^2, then $s+t = \{s_n + t_n\}_{n=1}^{\infty}$ is in l^2 and

$$\left[\sum_{n=1}^{\infty} (s_n+t_n)^2\right]^{1/2} \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}$$

Proof: By hypothesis the series

$\sum_{n=1}^{\infty} s_n^2$ and $\sum_{n=1}^{\infty} t_n^2$ converge.

By the above theorem the series

$\sum_{n=1}^{\infty} s_n t_n$ converges.

$\because (s_n+t_n)^2 = s_n^2 + 2 s_n t_n + t_n^2$

$\therefore$ By a known theorem,

$\sum_{n=1}^{\infty} (s_n+t_n)^2$ converges and

$$\sum_{n=1}^{\infty} (s_n+t_n)^2 = \sum_{n=1}^{\infty} s_n^2 + 2\sum_{n=1}^{\infty} s_n t_n + \sum_{n=1}^{\infty} t_n^2.$$

Using the Schwartz inequality to the second terms on the right, we get.

$$\sum_{n=1}^{\infty} (s_n+t_n)^2 \le \sum_{n=1}^{\infty} s_n^2 + 2\left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2} + \sum_{n=1}^{\infty} t_n^2$$

(i.e) $$\sum_{n=1}^{\infty} (s_n+t_n)^2 \le \left[\left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} + \left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}\right]^2$$

Taking the square root on both sides

$$\left[\sum_{n=1}^{\infty}(s_n+t_n)^2\right]^{1/2} \le \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2}+\left(\sum_{n=1}^{\infty} t_n^2\right)^{1/2}$$

Hence proved.

Definition:-

If $S=\{s_n\}_{n=1}^{\infty}$ is an element of l^2 we define $\|s_n\|_2$ called the norm of S, as

$$\|s\|_2=\left[\sum_{n=1}^{\infty} s_n^2\right]^{1/2}$$

Theorem: 6.

The norm for sequence in l^2 has the following properties

$\|s_2\| \ge 0 \ (s \in l^2)$ ------> (1)

$\|s_2\| = 0$ iff $s=\{0\}_{n=1}^{\infty}$ -----> (2)

$\|cs\|_2 = |c| \cdot \|s\|_2 \ (c \in R;\ s \in l^2)$ ----> (3)

$\|s+t\|_2 \le \|s\|_2+\|t\|_2 \ (s, t \in l^2)$ ------> (4)

Proof :-

(1) clearly $\|s_2\| = \left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2} \ge 0$ thus (1) is proved

(2) If $\|s_2\|=0$, then $\left(\sum_{n=1}^{\infty} s_n^2\right)^{1/2}=0$

$\Leftrightarrow \sum_{n=1}^{\infty} s_n^2 = 0$

$\Leftrightarrow s_n = 0 \ \forall n$

(i.e) $s = (s_1, s_2, \ldots\ldots s_n, \ldots\ldots)$

$= (0, 0, \ldots\ldots) = 0$

This proves (3)

(4) By Minkowski's inequality, we have

$$\left(\sum_{n=1}^{\infty}(s_n+t_n)^2\right)^{1/2} \leq \left(\sum_{n=1}^{\infty}s_n^2\right)^{1/2}+\left(\sum_{n=1}^{\infty}t_n^2\right)^{1/2}$$

(i.e) $\|s+t\|_2 \leq \|s\|_2+\|t\|_2$

8th

4.1 Limit of a function on a Real Line

Defn:

Let $a \in \mathbb{R}$ and f be a real valued function whose domain includes all points in some open interval $(a-h, a+h)$ except possibly the point a itself. We say that $f(x)$ approaches L $(L \in \mathbb{R})$ as x approaches a if given $\epsilon>0$ $\exists$ $a \ \delta>0$

$|f(x)-L|<\epsilon$ $(0<|x-a|<\delta)$

In this case we write, $\lim_{x\to a} f(x)=L$ (or)

$f(x)\to L$ as $x\to a$.

Right hand and Left hand limits (one sided limits)

1. $f(x)$ is said to approach L as x approaches a from the right if for given $\epsilon>0$ $\exists$ $\delta>0$ $\ni$:

$|f(x)-L|<\epsilon$, $(a<x<a+\delta)$

In this case we write

$\lim_{x\to a^+} f(x)=L$ (L is called the right hand limit of f at a)

2. We say that $f(x)$ approaches L as x approaches a from the left if given $\epsilon>0$ $\exists$ a $\delta>0$ $\ni$:

$|f(x) - M| < \epsilon \quad (a - \delta < x < a)$

We write $\lim_{x \to a} f(x) = M$ and m is called left hand limit of f at a.

3. The limit of a function exists if both one side limit exist and are equal.

(i.e) $\lim_{x \to a} f(x) = l$ iff $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$

Defn:

The function $f(x)$ is said to approaches l as x approches infinity, if for given $\epsilon > 0$ $\exists\, M \in R \ni |f(x) - l| < \epsilon \quad (x > M)$

we write $\lim_{x \to \infty} f(x) = l$ (or) $f(x) \to l$ as $x \to \infty$

Algebra of limits

Theorem 1: 2nd

The limit of a sum is equal to the sum of the limits.

Proof: Let $\lim_{x \to a} f(x) = l$ and

$\lim_{x \to a} g(x) = M$

Then we have to prove that

$\lim_{x \to a} [f(x) + g(x)] = l + M$

(i.e) we have to show that for any pre-assigned the number ϵ, a number δ can be determined $\ni$:

$$|(f(x)+g(x)) - (L+M)| < \epsilon$$

whenever $0<|x-a|<\delta$

By hypothesis $\lim_{x\to a} f(x)=L$ so that

$|f(x)-L| < \epsilon/2$ whenever $0<|x-a|<\delta_1 \ldots\ldots(1)$

$|||^{ly}$ $|g(x)-M| < \epsilon/2$ whenever $0<|x-a|<\delta_2 \ldots\ldots(2)$

choosing $\delta = \min(\delta_1, \delta_2)$ it follows from

(1) & (2),

$$|(f(x)+g(x)) - (L+M)| = |(f(x)-L) + g(x)-M)|$$
$$\leq |f(x)-L| + |g(x)-M|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

whenever $0<|x-a|<\delta$.

$|||^{ly}$ we can prove, (i.e) $\lim_{x\to a} [f(x)+g(x)] = L+M$.

$\lim_{x\to\infty} [f(x) - g(x)] = L - M$

Theorem : 8

the limit of a product is equal to the product of the limits.

Proof: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

we have to p.T $\lim_{x\to a} f(x).g(x) = L.M$

(i.e) to prove $|f(x)g(x) - LM| < \epsilon$ whenever $0<|x-a|<\delta$

Now $|f(x)g(x) - LM| = |f(x)g(x) - Lg(x) + Lg(x) - LM|$

$\leq |f(x)g(x) - Lg(x)| + |L.g(x) - LM|$

$\leq |g(x)||f(x)-L| + |L||g(x)-M|$

By hypothesis $\lim_{x\to a} g(x) = M$. So that $g(x)$ is bounded in the neighbourhood of $x = a$.

Hence $|g(x)| < k$ for all values of x s.t. $0 < |x-a| < \delta'$.

Thus $|f(x)g(x) - LM| < k \cdot |f(x) - L| + |L|\,|g(x) - M|$

since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to \infty} g(x) = M$.

corresponding to any $\epsilon > 0$, we can find a +ve number $\delta < \delta'$ s.t.

$|f(x) - L| < \epsilon/2k$ and

$|g(x) - M| < \dfrac{\epsilon}{2(|L|)}$ whenever $0 < |x-a| < \delta$

Thus $|f(x)g(x) - LM| < k \cdot \dfrac{\epsilon}{2k} + |L| \dfrac{\epsilon}{2|L|} < \epsilon$

whenever $0 < |x-a| < \delta$

Hence $\lim_{x\to a} f(x) \cdot g(x) = LM$

Hence proved.

Theorem: 9

The limit of a quotient is equal to the quotient of the limits provided that the limit of the denominator is not zero.

Proof:- Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M (\neq 0)$

Now, $\left|\dfrac{f(x)}{g(x)} - \dfrac{L}{M}\right| = \left|\left(\dfrac{f(x)}{g(x)} - \dfrac{f(x)}{M}\right) + \left(\dfrac{f(x)}{M} - \dfrac{L}{M}\right)\right|$

$\dfrac{M f(x) - f(x)g(x)}{M g(x)}$

We have to P.T

$\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ i.e to prove

$\left|\dfrac{f(x)}{g(x)} - \dfrac{L}{M}\right| < \epsilon$ whenever $0 < |x-a| < \delta$.

$$= \left| \frac{f(x)}{Mg(x)} [M - g(x)] + \frac{1}{M} [f(x) - L] \right|$$

$$\le \frac{|f(x)|}{|M|\,|g(x)|} |M - g(x)| + \frac{1}{|M|} |f(x) - L| \rightarrow (1)$$

By hypothesis $\lim_{x \to a} g(x) = M$.

Hence the functions f and g are surely bounded in the neighbourhood of $x = a$.

Let l be the upperbound of $|f|$ and m be the lower bound for $|g|$

so, that, $|f(x)| < l$ and $|g(x)| > m$.

$\therefore$ (1) becomes,

$$\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| \le \frac{l}{m|M|} |M - g(x)| + \frac{1}{|M|} |f(x) - L| \rightarrow (2)$$

Since $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ corresponding to any $\epsilon > 0$ we can find numbers δ_1 and δ_2

$\ni$: $|f(x) - L| < |M| \epsilon/2$ whenever $0 < |x - a| < \delta_1$

and $|g(x) - M| < \frac{m \cdot |M|}{l} \epsilon/2$ whenever $0 < |x - a| < \delta_2$

choose $\delta = \min(\delta_1, \delta_2)$ from (2) we get,

$$\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| \le \epsilon/2 + \epsilon/2 = \epsilon \text{ whenever } 0 < |x - a| < \delta$$

Hence $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$.

Defn:

If f is a real valued function on an interval $J \subset R$ we say that f is non-decreasing on J if $f(x) \le f(y)$ $(x < y;\ x, y \in J)$ and if $f(x) \ge f(y)$ $(x < y,\ x, y \in J)$ we say that f is non increasing on J.

f is said to be monotone if f is either non decreasing or non increasing.

Theorem: 10

Let f be a non-decreasing function on the bounded open interval (a, b). If f is bounded above on (a, b) then $\lim_{x \to b^-} f(x)$ exists. Also if f is bounded below on (a, b) then $\lim_{x \to a^+} f(x)$ exists.

Proof:-

Let f be bounded above and non-decreasing on (a, b)

Let $M = \text{l.u.b } f(x),\ x \in (a, b)$

Given $\epsilon > 0$, the number $M - \epsilon$ is not an u.b for the range of f

Hence $\exists\ y \in (a, b) \ni: f(y) > M - \epsilon$

Let $\delta = b - y$ then $f(b - \delta) > M - \epsilon$

$\therefore f$ is non decreasing we have

$$f(x) > M - \epsilon \quad (b - \delta < x < b)$$

Hence $f(x) \le M\ \forall x \in (a, b)$

We obtain $|f(x) - M| < \epsilon,\ b - \delta < x < b$

$$\Rightarrow \lim_{x \to b^-} f(x) = M.$$

If f is bounded below we can by a similar argument prove that

$\lim_{x \to a^+} f(x) = m$, where $m = g.l.b\ f(x),\ x \in (a,b)$

Theorem : 11

Let f is a monotone ~~non. increasing~~ fn on the open interval (a,b) and $c \in (a,b)$ then $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exist

proof: Suppose that f is a non-decreasing on (a,b).

Let us choose $\delta > 0 \ni$: the bounded open interval $(c-\delta, c+\delta)$ is contained in (a,b) then the values of f on the open interval $(c-\delta, c)$ are bounded above by $f(c)$ and hence by theorem $\lim_{x \to c^-} f(x)$ exists

III^{ly} the values of f on the open interval $(c, c+\delta)$ are bounded below by $f(c)$.

Hence by the same theorem $\lim_{x \to c^+} f(x)$ exists

If f is non-increasing we should use theorem 5 to obtain the result.

Defn: The real valued fn. f on the interval $J \subset R$ is said to be strictly increasing if

$f(x) < f(y)\ (x < y;\ x, y \in J)$

If $f(x) > f(y)\ (x < y;\ x, y \in J)$ f is said to be strictly decreasing.

Problems:

Prove directly from defn: of limit that

1. $\lim_{x\to 1} (x^2+4x) = 5$

Soln: Here $f(x) = x^2+4x$, $L=5$, $a=1$

Given $\epsilon > 0$ we must find $\delta > 0 \ni$

$|(x^2+4x)-5| < \epsilon \quad (0<|x-1|<\delta)$ ---------→(1)

Now, $|x^2+4x-5| = |(x-1)(x+5)| = |x-1|\,|x+5|$ -----→(2)

First let us choose $\delta = 1$

If $|x-1| < \delta$ then we have

$|x-1| < 1$

$\Rightarrow x \in (a-\delta, a+\delta)$

$x \in (0, 2)$ and

$\therefore x+5 \in (5, 7)$

Hence if $|x+5| < 7$ and if $|x-1| < \delta < 1$

we have $|x-1|\,|x+5| < 7\delta$

Let $\delta = \min(1, \epsilon/7)$

Then $|x-1|\,|x+5| < 7\delta \le \epsilon$ whenever $|x-1| < \delta$

which implies (1)

$\therefore$ Given $\epsilon > 0$ we have found $\delta = \min(1, \epsilon/7)$

for which (1) is true and this proves

$\lim_{x\to 1} x^2+4x = 5$

2. P.T $\lim_{x\to\infty} \left(\frac{1}{x^2}\right) = 0$

Sol: Given $\epsilon > 0$ we must find $M \in \mathbb{R} \ni$:

$\left|\frac{1}{x^2} - 0\right| < \epsilon \quad (x > M)$ ------→(1)

$\Rightarrow 1/x < \sqrt{\epsilon} \quad (x > M)$

Hence if we take $M = \frac{1}{\sqrt{\epsilon}}$ (1) will hold.

less than ∞ ⇒ converges

3. Evaluate $\lim_{x\to 0^+} x e^{1/x}$

Sol: $\lim_{x\to 0^+} x e^{1/x}$

$= \lim_{h\to 0} (0+h)\, e^{\frac{1}{0+h}}$ $(h>0)$

$= \lim_{h\to 0} h\left[1+\frac{1}{h}+\frac{1}{L2}\cdot\frac{1}{h^2}+\cdots\right]$ $(h>0)$

$= \lim_{h\to 0} \left[h+1+\frac{1}{L2}\cdot\frac{1}{h}+\cdots\right]$ $(h>0)$

$= \infty$

4. Test whether $\lim_{x\to 0^+} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$ exists.

Sol: Now,

$$\lim_{x\to 0^+} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} = \lim_{h\to 0} \frac{e^{1/0+h}-e^{-1/0+h}}{e^{1/0+h}+e^{-1/0+h}} \quad (h>0)$$

$$= \lim_{h\to 0} \frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}$$

$$= \lim_{h\to 0} \frac{1-e^{-2/h}}{1+e^{-2/h}} = 1 \quad \left[\because \lim_{h\to\infty} e^{1/h} = 0\right]$$

Next let us consider,

$$\lim_{x\to 0^-} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} = \lim_{h\to 0} \frac{e^{\frac{1}{0-h}}-e^{-\frac{1}{0-h}}}{e^{\frac{1}{0-h}}+e^{-\frac{1}{0-h}}} \quad (h>0)$$

$$= \lim_{h\to 0} \frac{e^{-1/h}-e^{+1/h}}{e^{-1/h}+e^{1/h}} \quad (h>0)$$

$$= \lim_{h\to 0} \frac{e^{-2/h}-1}{e^{-2/h}+1} \quad (h>0)$$

$= -1$

$\because \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

The limit does not exist.

prove that $\lim_{x\to 2} \sqrt{x^2-4} = 0$

5

proof: Given $\epsilon > 0$ we have to find $\delta > 0 \ni$

$|\sqrt{x^2-4} - 0| < \epsilon$ ----> (1) Here $a=2$, $L=0$

when $|x-2| < \delta$

Now $|\sqrt{x^2-4} - 0| = |\sqrt{(x+2)(x-2)}| \Rightarrow \sqrt{x+2}\,\sqrt{x-2}$

$|x-2| < \delta \le 1 \Rightarrow x \in (a-\delta, a+\delta)$

$\Rightarrow x \in (1,3)$

$\Rightarrow x+2 \in (3,5)$

$\Rightarrow \sqrt{x+2} < \sqrt{5}$

so $|\sqrt{x^2-4}| < \sqrt{5}\,\sqrt{\delta} < \epsilon$ if we choose $\delta = \min\left(1, \frac{\epsilon^2}{5}\right)$

Hence (1) holds for the above choice of δ.

6 P.T $\lim_{x\to 1} \sqrt{x+3} = 2$

sol: Whenever $|x-1| < \delta \le 1$ we have to P.T $|f(x)-L| < \epsilon$

(i.e) to prove $|f(x)-L| < \epsilon$ ------> (1)

consider,

$$|\sqrt{x+3} - 2| = \frac{|(\sqrt{x+3}-2)(\sqrt{x+3}+2)|}{|\sqrt{x+3}+2|}$$

If $\delta \le 1$ then $|x-1| < \delta$

$x \in (0,2)$

Hence $\sqrt{x+3} + 2 > \sqrt{3} + 2$

$$\text{So } \frac{1}{\sqrt{x+3}+2} < \frac{1}{\sqrt{3}+2}$$

$$\text{Thus } |\sqrt{x+3} - 2| = \frac{|x-1|}{|\sqrt{x+3}+2|}$$

$$< \frac{\delta}{\sqrt{3}+2}$$

If we choose $\delta = \min(1, \epsilon(\sqrt{3}+2))$ then

$|\sqrt{x+3} - 2| < \epsilon$.

4.2 Metric Spaces 5th

Defn: A metric on a non-empty set M is map $\rho : M \times M \to R$

(i) $\rho(x, x) = 0 \quad (x \in M)$

(ii) $\rho(x, y) > 0 \quad (x, y \in M, x \neq y)$

(iii) $\rho(x, y) = \rho(y, x) \quad (x, y \in M)$

(triangle inequality)

If ρ is a metric for M, then the ordered pair (M, ρ) is called a metric space.

Example: 1 (usual Metric on R)

Let $M = R$. Define $\rho : M \times M \to R$ by

$\rho(x, y) = |x - y|$, $x, y \in M$

this is clearly a metric on M. This metric is known as the usual metric or absolute value metric on R. We denote the resulting metric space (R, ρ) by R'.

Example: 2 (Discrete Metric Space).

Let M be any non-empty set

Define $\rho : M \times M \to R$ by $\rho(x, y) = 1$ if $x \neq y$

$= 0$ if $x = y$

From the defn. we see that,

(i) $\rho(x,x)=0$. $\forall x \in M$. $\rho(x,y)=1>0$. $\forall x,y \in M$ with $x \neq y$.

(ii) $\rho(x,y)=1=\rho(y,x)$ $x,y \in M$, $x \neq y$

Let us verify triangle inequality.

Let $x=y=z$ then $\rho(x,y)=0$, $\rho(x,z)=0$ and $\rho(y,z)=0$ and so $\rho(x,y) \leq \rho(x,z)+\rho(y,z)$

Let any two of them be same. Say $x=y \neq z$ then $\rho(x,y)=0$ $\rho(x,z)=1$ and $\rho(z,y)=1$

so that $0=\rho(x,y) \leq \rho(x,z)+\rho(z,y)$

$= 1+1=2$

Let $x \neq y \neq z$. Then $\rho(x,y)=1$, $\rho(x,z)=1$, $\rho(z,y)=1$

$\therefore \rho(x,y) \leq \rho(x,z)+\rho(z,y)$

$1 \leq 1+1=2$

Thus (M,ρ) is a metric space and it is known as discrete metric space. This metric space is denoted by R_d.

Inequalities:

(1) Holder's Inequality:

If $p>1$ and q is $\ni: \frac{1}{p}+\frac{1}{q}=1$. then

$$\sum_{i=1}^{n}|a_ib_i| \leq \left[\sum_{i=1}^{n}|a_i|^p\right]^{1/p}\left[\sum_{i=1}^{n}|b_i|^q\right]^{1/q}$$

where $a_1, a_2, \ldots\ldots a_n$ and $b_1, b_2, \ldots\ldots b_n$ are real numbers.

(2) Cauchy - Schwarz Inequality:

Putting $p=q=2$ in Holder's inequality we get

$$\sum_{i=1}^{n}|a_ib_i| \leq \left[\sum_{i=1}^{n}|a_i|^2\right]^{1/2}\left[\sum_{i=1}^{n}|b_i|^2\right]^{1/2}$$

This is called Cauchy - Schwarz inequality.

(3) Minkowski's inequality:

If $p \geq 1$ and $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ are real numbers. Then

$$\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p}$$

Example: (3) (n-dimensional Euclidean space)

Let us fix $n \in I$.

If $x = (x_1, x_2, \ldots x_n)$ and

$y = (y_1, y_2, \ldots y_n)$ are two ordered

n-tuples of real numbers. define

$$P(x,y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2\right]^{1/2}$$

[For $n=2$, $P(x,y)$ is the usual distance formula for the points in the Cartesian plane].

clearly (i) $P(x,x) = 0$

(ii) $P(x,y) > 0$ $x \neq y$ and

(iii) $P(x,y) = P(y,x)$

Let us show that P satisfies the triangle inequality. thus if $z = (z_1, z_2, \ldots z_n)$

we have to show that $P(x,y) \leq P(x,z) + P(z,y)$

Let $x_k - z_k = a_k$ and

$z_k - y_k = b_k$ for $k = 1, 2, \ldots n$

then $$P(x,z) = \left[\sum_{k=1}^{n} (x_k - z_k)^2\right]^{1/2}$$

$$= \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}.$$

$$P(z,y) = \left[\sum_{k=1}^{n} (z_k - y_k)^2\right]^{1/2}$$

$$= \left[\sum_{k=1}^{n} b_k^2 \right]^{1/2}$$

and $\rho(x,y) = \left[\sum_{k=1}^{n} (a_k + b_k)^2 \right]^{1/2}$ $\left[\because a_k + b_k = x_k - y_k \right]$

Now $\left[\sum_{k=1}^{n} (a_k+b_k)^2 \right]^{1/2} \le \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2 \right)^{1/2}$

Follows from the "minkowski's inequality".

thus ρ satisfies all conditions for a metric.

we denote this metric space by R^n. It is called Euclidean n-space.

Example: 4 (l^∞ space)

Let l^∞ denote the set of all bounded sequences of real numbers.

If $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ are points in l^∞.

define $\underset{1 \le n < \infty}{\text{l.u.b}} |x_n - y_n|$ we can easily prove that ρ satisfies the first three conditions for a metric space.

For triangular inequality,

Let $z = \{z_n\}_{n=1}^{\infty}$ also be a point in l^∞.

For any $k \in I$,

$$|x_k - y_k| = |x_k - z_k + z_k - y_k|$$
$$\le |x_k - z_k| + |z_k - y_k|$$
$$\le \underset{1 \le n < \infty}{\text{l.u.b}} |x_n - z_n| + \underset{1 \le n < \infty}{\text{l.u.b}} |z_n - y_n|$$

$\therefore |x_k - y_k| \le \rho(x,z) + \rho(z,y)$, $k \in I$

From this we have,

$\underset{1 \le k < \infty}{\text{l.u.b}} |x_k - y_k| \le \rho(x,z) + \rho(z,y)$

(i.e) $\rho(x,y) \le \rho(x,z) + \rho(z,y)$

Example: 5 (l^2 space)

Let l^2 be the class of all sequences $S = \{S_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} S_n^2 < \infty$. Further $\|S\|_2 = \left[\sum_{n=1}^{\infty} S_n^2\right]^{1/2}$. For $x, y \in l^2$, define

$\rho(x,y) = \|x-y\|_2$

we have (1) $\|x\|_2 \geq 0$ $\therefore d(x,y) = \|x-y\|_2 \geq 0$

(2) $\|x\|_2 = 0 \iff x = \{0\}_{n=1}^{\infty}$ $d(x,y) = \|x-y\|_2 = 0 \iff x=y$

For triangular inequality,

Let $z = \{z_n\}_{n=1}^{\infty}$ also be a point in l^{∞},

for any $k \in I$.

$$|x_k - y_k| = |x_k - z_k + z_k - y_k|$$

$$\leq |x_k - z_k| + |z_k - y_k|$$

$$\leq \underset{1 \leq n < \infty}{\text{l.u.b}} |x_n - z_n| + \underset{1 \leq n < \infty}{\text{l.u.b}} |z_n - y_n|$$

$\therefore |x_k - y_k| \leq \rho(x,z) + \rho(z,y)$, $k \in I$

From this we have,

$$\underset{1 \leq k \leq \infty}{\text{l.u.b}} |x_k - y_k| \leq \rho(x,z) + \rho(z,y)$$

(i.e) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$

(3) $\rho(x,y) = \|x-y\|_2 = \|(-1)(y-x)\|_2$

$$= |(-1)| \, \|(y-x)\|_2$$

$$= \|y-x\|_2$$

(4) Also for $x, y, z \in l^2$

$$\rho(x,y) = \|x-y\|_2$$

$$= \|x-z+z-y\|_2$$

$\leq \|x-z\|_2 + \|z-y\|_2$

$\leq \rho(x,z) + \rho(z,y)$

Hence ρ is a metric for ℓ^2

$\therefore (\ell^2, \rho)$ is a metric space and it is denoted by ℓ^2.

Here $\|S\|_2$ is called the norm of the metric space.

Example: 6

Let (M,d) be a metric space. Define d_1 by $d_1(x,y) = \min\{1, d(x,y)\}$ then d_1 is a metric for M.

Soln: Let $x,y \in M$

(1) d is metric on M

$\Rightarrow d(x,y) \geq 0$

$\Rightarrow \min\{1, d(x,y)\} \geq 0$

$\Rightarrow d_1(x,y) \geq 0$

(2) $d_1(x,y) = 0 \Leftrightarrow \min\{1, d(x,y)\} = 0$

$\Leftrightarrow d(x,y) = 0$

$x = y$

(3) $d_1(x,y) = \min\{1, d(x,y)\}$

$= \min\{1, d(y,x)\}$

$= d_1(y,x)$

(4) Let $x,y,z \in M$. We now prove the triangle inequality.

(i.e) $d_1(x,y) + d_1(y,z) \geq d_1(x,z)$

Now $d_1(x,z) = \min\{1, d(x,z)\} \leq 1$

So if $d_1(x,y) = 1$ or $d_1(y,z) = 1$

Certainly triangle inequality holds.

Suppose $d_1(x,y) < 1$ and $d_1(y,z) < 1$

~~Thus d_1.~~

Then $d_1(x,y) = d(x,y)$ and

$d_1(y,z) = d(y,z)$

According $d_1(x,y) + d_1(y,z) = d(x,y) + d(y,z)$

$\geq d(x,z)$ ($\because d$ is metric)

$= d_1(x,z)$

Thus d_1 is also a metric on M.

Theorem : 12

Let (M, ρ) be a metric space and let a be a point in M. Let f and g be real valued functions (i.e. functions with range in R' with absoluted value metric) whose domains are subsets of M. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = N$ then,

(i) $\lim_{x \to a} [f(x) + g(x)] = L + N$

(ii) $\lim_{x \to a} [f(x) - g(x)] = L - N$

(iii) $\lim_{x \to a} [f(x) g(x)] = LN$

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{N}$ $(N \neq 0)$

Proof:

We shall prove (iii)

(i.e) $\lim_{x \to a} [f(x) g(x)] = LN$

$\because \lim_{x \to a} g(x) = N$, we have for some $\delta_1 > 0$

$|g(x) - N| < 1$ $(0 < \rho(x,a) < \delta_1)$

$\Rightarrow |g(x)| = |g(x) - N + N|$

$\leq |g(x) - N| + |N|$

$< 1 + |N| = Q$ say

$(0 < \rho(x,a) < \delta_1)$

Now $f(x)g(x) - LN$

$= f(x)g(x) - Lg(x) + Lg(x) - LN$

$= g(x)[f(x) - L] + L[g(x) - N]$

$\therefore |f(x)g(x) - LN| \leq |g(x)||f(x) - L| + |L||g(x) - N|$

$\leq Q|f(x) - L| + |L||g(x) - N| \dashrightarrow (1)$

if $0 < \rho(x,a) < \delta_1$

Given $\epsilon > 0$, $\exists\ \delta_2 > 0 \ni$:

$Q|f(x) - L| < \epsilon/2$, $0 < \rho(x,a) < \delta_2 \dashrightarrow (2)$

and also $\exists\ \delta_3 > 0 \ni: |L||g(x) - N| < \epsilon/2 \dashrightarrow (3)$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$

Then from (1), (2) & (3), it follows that

$|f(x)g(x) - LN| < \epsilon$, $[0 < \rho(x,a) < \delta]$

$\Rightarrow \lim_{x \to a} f(x)g(x) = LN$

Proof of (iv)

Since $\lim_{x \to a} g(x) = N \neq 0$

Let $\epsilon = \frac{|N|}{2}$, we can find a $\delta_1 > 0 \ni$:

$|g(x) - N| < \frac{|N|}{2}$, $0 < \rho(x,a) < \delta_1 \dashrightarrow (1)$

Now $|N| = |N - g(x) + g(x)|$

$\leq |N - g(x)| + |g(x)|$ $(0 < \rho(x,a) < \delta_1)$

$\Rightarrow |N| \leq \frac{|N|}{2} + |g(x)|$

$|g(x)| > \frac{|N|}{2}$, $0 < \rho(x,a) < \delta_1 \dashrightarrow (2)$

Consider,

$$\left|\frac{f(x)}{g(x)} - \frac{L}{N}\right| = \left|\frac{Nf(x) - Lg(x)}{Ng(x)}\right|$$

$$= \left|\frac{Nf(x) - LN + NL - Lg(x)}{Ng(x)}\right|$$

$$\leq \left|\frac{N[f(x) - L]}{Ng(x)}\right| + \left|\frac{L[g(x) - N]}{Ng(x)}\right|$$

$$\leq 2\frac{|f(x) - L|}{|N|} + \frac{2|L|}{|N|^2}|g(x) - N|$$

(using (2))

$$\leq 2\frac{|f(x) - L|}{|N|} + \frac{2|L|}{N^2}|g(x) - N| \dashrightarrow (3)$$

whenever $0 < P(x,a) < \delta_1$

Let $\epsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$ and

$\lim_{x \to a} g(x) = N \; (N \neq 0)$

we can find $\delta_2, \delta_3 > 0 \ni$:

$$|f(x) - L| < \frac{\epsilon |N|}{4}, \; 0 < P(x,a) < \delta_2 \dashrightarrow (4)$$

and $|g(x) - N| < \frac{\epsilon |N|^2}{4|L|}, \; 0 < P(x,a) < \delta_3 \dashrightarrow (5)$

Let us choose $\delta = \min(\delta_1, \delta_2, \delta_3)$

Then from (3), (4) & (5)

We find that $\left|\frac{f(x)}{g(x)} - \frac{L}{N}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

when $0 < P(x,a) < \delta$

This implies that,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{N} \quad (N \neq 0)$$

Defn:

A sequence of points in a metric space is a function from I into M. As with sequences of real numbers, a sequence of points in M is written as $\{a_n\}_{n=1}^{\infty}$

Defn:

Let (M, ρ) be a metric space and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of points in M.

We say that s_n approches L (where $L \in M$) as n approaches infinity if given $\epsilon > 0$ $\exists N \in I \ni$:

$$\rho(s_n, L) < \epsilon \quad (n \geq N)$$

In this case we write $\lim\limits_{n \to \infty} s_n = L$ (or) $s_n \to L$ as $n \to \infty$ and say that $\{s_n\}_{n=1}^{\infty}$ is convergent in M to the point L.

Defn: Cauchy sequence

Let (M, ρ) be a metric space and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of points in M. We say that $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence if given $\epsilon > 0$ $\exists N \in I \ni$:

$$\rho(s_m, s_n) < \epsilon \quad (m, n \geq N)$$

theorem: 13

Let (M, ρ) be a metric space. If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points in M then $\{s_n\}_{n=1}^{\infty}$ is cauchy.

Proof:-

Let $\{S_n\}_{n=1}^{\infty}$ be a convergent sequence of points in M.

Let $L = \lim_{n \to \infty} S_n$ (where $L \in M$)

Then given $\epsilon > 0$ there exists $N \in I \ni$;

$\rho(S_n, L) < \epsilon/2$, $(n \geq N)$

Hence if $m, n \geq N$

we have $\rho(S_m, S_n) \leq \rho(S_m, L) + \rho(L, S_n)$ (by Δ^{le} inequality)

$\leq \rho(S_m, L) + \rho(S_n, L)$

(By symmetry)

$\leq \epsilon/2 + \epsilon/2 = \epsilon$.

Hence $\{S_n\}_{n=1}^{\infty}$ is a cauchy sequence.

Remark:

The converse of the above result need not hold for all metric spaces. For some metric spaces there are cauchy sequences which are not convergent.

Consider the following example;

Let M be the set of all points (x, y) in the Euclidean plane $R^2 \ni$;

$x^2 + y^2 < 1$, with the R^2 metric used as metric for M.

The sequence $A = \left\{0, \frac{n}{n+1}\right\}_{n=1}^{\infty}$ is a Cauchy sequence of points in M but there is no $L \in M \ni$; A is convergent to L.

Hence the sequence A of points of M is not convergent in M.

Example: 7

For $x, y \in \mathbb{R}^n$ define $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

x8 & 9 See in books GV pages 4.19.

4.3 Limits in Metric Spaces:

Defn: Suppose that (M_1, ρ_1) and (M_2, ρ_2) are metric spaces, that $a \in M_1$ and that f is a function whose range is contained in M_2 and whose domain contains all $x \in M_1$ s.t.

$\rho_1(a, x) < \delta$ for some $\delta > 0$ except possibly $x = a$

We say that $f(x)$ approaches L (where $L \in M_2$ as x approaches a if given $\epsilon > 0$ $\exists\, \delta > 0$ s.t.

$\rho_2(f(x), L) < \epsilon$ $(0 < \rho_1(x, a) < \delta)$

In this case we write

$\lim_{x \to a} f(x) = L$ or $f(x) \to L$ as $x \to a$

Note:

If $(M_1, \rho_1) = (M_2, \rho_2) = \mathbb{R}$ then

$\rho_2(f(x), L) = |f(x) - L|$

$\rho_1(x, a) = |x - a|$

UNIT-V

CONTINUOUS FUNCTIONS ON METRIC SPACES

5.1. Functions continuous at a point on Real line.

Defn:

Let a be a point in a real line R^1 and let f be a real-valued functions whose domain contains all points of some open interval $(a-h, a+h)$ including a itself where $h>0$. 'f is said to be continuous at $a \in R^1$ if $\lim_{x \to a} f(x) = f(a)$

(or) f is continuous at $x=a$ if for every $\epsilon>0$, $\exists$ a $\delta>0$ $\ni: |f(x)-f(a)| < \epsilon$ for $0<|x-a|<\delta$.

Example:

(1): Consider $f(x) = \frac{\sin x}{x}$ $x \in R^1$ $x \neq 0$.

The fn is not defined at $x=0$ and hence is not contⁿ at $x=0$ eventhough.

$\lim_{x \to 0} \frac{\sin x}{x} = 1$ exists. So if we define g by $g(x) = \frac{\sin x}{x}$ $(x \neq 0)$ and $g(0)=1$, then g is contⁿ at $x=0$ as $\lim_{x \to 0} g(x) = g(0)$

(2) In some cases $\lim_{x \to 0} f(x)$ may not exist.

For example,

let $\psi(x)=1$ if x is rational

$\psi(x)=0$ if x is irrational.

$\psi(a)$ is defined for any $a \in R^1$ but $\lim_{x \to a} \psi(x)$ does not exist for any a.

To see this, assume the contrary

$\lim_{x \to a} \psi(x) = l$ for some $l \in R^1$.

Given $\epsilon = 1/3$ $\exists$ $\delta > 0$ $\ni$:

$|\psi(x) - l| < 1/3$ for $0 < |x-a| < \delta$

But in the interval $(a, a+\delta)$ (say)

there are both rational and irrational numbers.

If $x \in (a, a+\delta)$ is rational then,

$|1 - l| < 1/3 \Rightarrow 2/3 < l < 4/3$

If $x \in (a, a+\delta)$ is irrational then

$|0 - l| < 1/3 \Rightarrow -1/3 < l < 1/3$

a contradiction

Hence $\lim_{x \to a} \psi(x)$ does not exist at any point $x \in R^1$.

So ψ is not continuous at any point of its domain.

Problems:

1. If $f(x) = \sqrt{x^2 - 4}$ for $x \geq 2$ prove that $f(x)$ is cont^n at $x = 2$ finding a δ for a given ϵ.

Proof:- Given $|x-2| < \delta$ we have to P.T

$|\sqrt{x^2-4} - 0| < \epsilon$ (Here $f(2) = 0$)

$\Rightarrow |\sqrt{(x+2)(x-2)}| = \sqrt{x+2}\,\sqrt{x-2}$

$|x-2| < \delta \leq 1$

$\Rightarrow x \in (1,3) \Rightarrow x+2 \in (3,5)$

$\Rightarrow \sqrt{x+2} < \sqrt{5}$

so,

$|\sqrt{x^2-4}| < \sqrt{5}\,\sqrt{\delta} < \epsilon$

If we choose $\delta = \min(1, \frac{\epsilon}{5})$ we find (1) is true.

Hence $f(x)$ is continuous at $x=2$.

Ex 2: P.T if $g(x) = \sqrt{x}$, $(0 < x < \infty)$ then g is continuous at each point of $(0, \infty)$

Sol: Let $a \in (0, \infty)$

then given $|x-a| < \delta \leq 1$ we have to P.T

$|f(x) - f(a)| < \epsilon$ for any given $\epsilon > 0$.

$\Rightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$

Now, $|\sqrt{x} - \sqrt{a}| = \left| \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}} \right|$

$= \frac{|x-a|}{\sqrt{x} + \sqrt{a}}$

$< |x-a|$

$\because x > 0$.

So $|\sqrt{x} - \sqrt{a}| < \epsilon$ if we choose $\delta = \epsilon$

This proves that $f(x) = \sqrt{x}$ is conti at $x=a$.

Ex 3: If $f(x) = x$ $(-\infty < x < \infty)$ P.T f is contn at each point in R^1.

Sol: let $a \in R^1$.

then for $|x-a| < \delta$ and given $\epsilon > 0$ we have $|f(x) - f(a)| < \epsilon$ for any $|x-a| < \delta$.

thus any $\delta < \epsilon$ will serve the purpose.

Hence $f(x) = x$ is conti at $x=a$.

$\because$ a is arbitrary, $f(x)$ is conti at each point in R^1.

Theorem: 1

If the real valued functions f and g are continuous at $a \in R^1$, then so are $f+g$, $f-g$ and fg. If $g(a) \neq 0$, then $\frac{f}{g}$ is also cont$^{\underline{s}}$ at a

proof:

$\Rightarrow$ Since f and g are cont$^{\underline{s}}$ at a we have $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to a} g(x) = g(a)$$

But we have proved that,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= f(a) + g(a)$$

(i.e) $\lim_{x \to a} (f+g)(x) = (f+g)(a)$.

This proves that $f+g$ is continuous at a

|||^{ly} other results can be proved.

Theorem: 2

If f is continuous then so is $|f|$.

Proof:

Let f be cont$^{\underline{s}}$ at a point a of its domain;

then given $\epsilon > 0$, $\exists$ a $\delta > 0 \ni$;

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ ------$\to$(1)

But $||f(x)| - |f(a)|| \leq |f(x) - f(a)|$

(i.e) $||f|(x) - |f|(a)| \leq |f(x) - f(a)|$ ------$\to$(2)

From (1) & (2) we see that

$|x-a| < \delta \Rightarrow ||f|(x) - |f|(a)| < \epsilon$

$\Rightarrow |f|$ is cont$^{\underline{s}}$ at a.

Theorem: 3

Let f and g be real valued functions. If f is conti at a and if g is conti at $f(a)$, then the composite fn. gof is also conti at a.

Proof: Let $b = f(a)$

Since g is conti at b for a give $\epsilon > 0$, $\exists$ $\delta > 0 \ni : |y-b| < \delta$

$\Rightarrow |g(y) - g(b)| < \epsilon$ -------- $\rightarrow (1)$

Again since f is conti at a corresponding to δ (taking ϵ to be δ in this case) we can find $\eta > 0 \ni$:

$|x-a| < \eta \Rightarrow |f(x) - f(a)| < \delta$

(or) $|x-a| < \eta \rightarrow |f(x) - b| < \delta$ ------ $\rightarrow (2)$

Now (2) shows that if $|x-a| < \eta \Rightarrow$ $f(x)$ lies in the interval $(b-\delta, b+\delta)$ and so we may substitute $f(x)$ for y in (1).

Hence we get from (1) & (2),

$|x-a| < \eta \Rightarrow |g(f(x)) - g(f(a))| < \epsilon$

$\Rightarrow gof$ is conti at a.

Hence proved

5.2 Reformulation:

By the defn of continuity of f at a, we have for any $\epsilon > 0$ $\exists$ $\delta > 0 \ni : |f(x) - f(a)| < \epsilon$ whenever $0 < |x-a| < \delta$.

The inequality $|f(x) - f(a)| < \epsilon$ holds for $x = a$ also. Thus it is enough we write $|x-a| < \delta$ instead of $0 < |x-a| < \delta$

Theorem:

The real-valued fn. f is conti at $a \in R'$ iff given $\epsilon > 0$ $\exists$ a $\delta > 0 \ni : |f(x) - f(a)| < \epsilon$ $(|x-a| < \delta)$

Remark:

According to above theorem we see that f is conti at a if for given $\epsilon > 0$ $\exists$ $\delta > 0 \ni :$ if the distance from x to a is less than ϵ then the distance from $f(x)$ to $f(a)$ will be less than ϵ.

Defn:

If $a \in R'$ and $r > 0$ we define $B[a;r]$ to be the set of all $x \in R'$ whose distance from a is less than r.

(i.e) $B[a;r] = \{x \in R' \mid |x-a| < r\}$

We call $B[a;r]$ the open ball of radius r about a

Theorem: 4

The real valued function f is conti at $a \in R'$ iff the inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a, \delta]$ about a.

Proof: We prove that f is conti iff for given $\epsilon>0$, there exists $\delta>0 \ni$:

$$f^{-1}(B[f(a);\epsilon]) \supset B[a,\delta]$$

We know that f is conti at a iff for $\epsilon>0$ $\exists\ \delta>0 \ni: |x-a|<\delta \Rightarrow |f(x)-f(a)|<\epsilon$

Hence $x \in B[a,\delta] \Rightarrow f(x) \in B[f(a);\epsilon]$

$\Rightarrow x \in f^{-1} B[f(a);\epsilon]$

Hence $B[a;\delta] \subset f^{-1}[B[f(a);\epsilon]]$

Hence proved.

Continuity and convergence:

The seq. $\{x_n\}_{n=1}^{\infty}$ converges to a iff given $\epsilon>0$ $\exists\ N \in I \ni: x_n \in B[a;\epsilon]$ $(n \geq N)$

(ie) given any open ball B about a, all but a finite number of x_n are in B.

Theorem: 5

The real valued function f is conti at $a \in R^1$ and only if whenever $\{x_n\}_{n=1}^{\infty}$ is a seq of real numbers converging to a then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$. i.e f is conti at a iff

$$\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = f(a) \text{ ------} \rightarrow (1)$$

Proof:-

Suppose that f is continuous at then we shall prove that (1) holds.

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence of real numbers converging to a.

then $f(x_n)$ will be defined for sufficiently large n.

We must show that $\lim_{n \to \infty} f(x_n) = f(a)$

(i.e) given $\epsilon > 0$ we must find $N \in I \ni$:

$f(x_n) \in B[f(a); \epsilon] \quad (n \geq N) \dashrightarrow (2)$

Since f is cont$\underline{n}$ at a $\exists$ $\delta > 0 \ni$:

$f(x) \in B[f(a); \epsilon], \quad x \in B[a; \delta] \longrightarrow (3)$

Further since $\lim_{n \to \infty} (x_n) = a$, $\exists N \in I$

$\ni$: $x_n \in B[a; \delta]$ when $n \geq N \dashrightarrow (4)$

For this N, condition (2) follows from (3) & (4)

Conversely suppose eqn (1) holds.

We must P.T f is cont$\underline{n}$ at a.

Let us assume the contrary

then by theorem (4) for some $\epsilon > 0$ the inverse image under f of $B = B[f(a); \epsilon]$ contains no open ball about a.

In particular

$f^{-1}(B)$ does not contain $B[a; 1/n]$ for any $n \in I$.

thus for each $n \in I$, there is a point $x_n \in B[a; 1/n] \ni$:

$f(x_n) \notin B$

$\Rightarrow |x_n - a| < 1/n$ but

$|f(x_n) - f(a)| \geq \epsilon$

This condradicts eqn (1)

Hence f is conti at $x = a$

Hence proved.

5.3 Functions continuous on metric spaces.

Defn:

Let (M, ρ) be a metric space. If $a \in M$ and $r > 0$ then $B[a; r]$ is defined to be the set of all points in M whose distance to a is less than r.

(i.e) $B[a; r] = \{x \in M / \rho(x, a) < r\}$ we call $B[a; r]$ the open ball of radius r about a.

Example: 1

If $M = R_d$, the real line with discrete metric, and if a is any point in R_d, then $B[a; 1] = \{a\}$. For the only point in R_d whose distance to a is less than 1 is a itself.

Example: 2

Consider the metric space $(R, d) = R'$ whose d is the usual metric. Find the open balls in R'.

Soln: Let $a \in R'$

Then $B[a; r] = \{x \in R' / d(a, x) < r\}$

$= \{x \in R' / |a - x| < r\}$

$= \{x \in R' / a - r < x < a + r\}$

$= (a - r, a + r)$

(i.e) open balls in R' are open intervals.

Example: 3

If M is the closed interval $[0,1]$ with absolute value metric then $B[1/4 ; 1/2]$ is the interval $[0 ; 3/4]$.

[Note that points in R^1 to the left of at 0 are not in M].

Example 4:

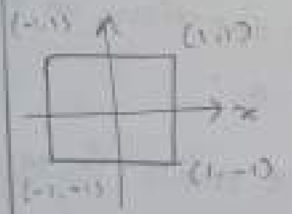
Consider the metric space (R^2, d) where d is the metric defined by $d[(x_1,y_1),(x_2,y_2)] = \max\{|x_1-x_2|, |y_1-y_2|\}$. Find $B[(0,0),1]$.

Sol:- $B[(0,0),1] = \{(x,y) \in R^2 / d[(0,0)(x,y)] < 1\}$

$= \{(x,y) \in R^2 / \max\{|0-x|, |0-y|\} < 1\}$

$= \{(x,y) \in R^2 / \max(|x|, |y|) < 1\}$

$= \{(x,y) \in R^2 / |x| < 1, |y| < 1\}$



= Set of all points inside the square with vertices $(1,1)(-1,1)(-1,-1)$, $(1,-1)$

Defn: continuity:

Let (M_1, ρ_1) and (M_2, ρ_2) be the metric spaces. Let $a \in M_1$ and f be any function whose range is contained in M_1 and whose domain contains some open ball $B(a;h)$. The function f is continuous at $a \in M_1$ if $\lim_{x \to a} f(x) = f(a)$

Theorem: 6

The function f is continuous at $a \in M_1$ iff any one (and hence all) of the following conditions hold.

(i) Given $\epsilon > 0$, $\exists\ \delta > 0 \ni \rho_2\ [f(x), f(a)] < \epsilon$, $(\rho_1(x,a) < \delta)$

(ii) the inverse image under f of any open ball $B[f(a); \epsilon]$ about $f(a)$ contains an open ball $B[a, \delta]$ about a.

(iii) whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1 converging to a, then the seq. $\{f(x_n)\}_{n=1}^{\infty}$ of points of M_2 converges to $f(a)$.

Proof:

Let us prove (iii) only. (The proofs of (i) and (ii) are similar to those corresponding to continuous functions on R^1).

Let $f: M_1 \to M_2$ be conti at a.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in M_1 converging to a.

Since f is continuous at a given $\epsilon > 0\ \exists : \delta > 0 \ni$;

$$\rho_2\ [f(x), f(a)] < \epsilon \text{ ---------} \to (1)$$

$$\text{whenever } \rho_1(x, a) < \delta$$

Since $x_n \to a$ we can find a positive integer $N \ni: \rho_1(x_n, a) < \delta\ \forall n \geq N \text{ --------} \to (2)$

by (1) it follows that.

$$\rho_2\ [f(x_n), f(a)] < \epsilon,\ \forall n \geq N$$

$$\Rightarrow f(x_n) \to f(a),\ n \to \infty$$

Conversely:

Suppose that every sequence $\{x_n\}$ in M_1 converging to a, the sequence $\{f(x_n)\}$ in M_2

converging to $f(a)$. then we have to prove that f is conti at a. Suppose that f is not conti at a. then for $\forall \epsilon > 0$ and for every $\delta > 0$ there is a point $x \in M_1 \ni$:

$\rho_2(f(x), f(a)) \geq \epsilon$ whenever $\rho_1(x, a) < \delta$

Let $\delta = 1$, then there is a point $x_1 \in M_1 \ni$:

$\rho_2(f(x_1), f(a)) \geq \epsilon$, $\rho_1(x_1, a) < 1$.

choose $\delta = 1/2$ then we can find a point $x_2 \in M_1 \ni$:

$\rho_2(f(x_2), f(a)) \geq \epsilon$

$\rho_1(x_2, a) < 1/2$

continuing in this manner if we choose $\delta = 1/n$ there is $x_n \in M_1 \ni$:

$\rho_2(f(x_n), f(a)) \geq \epsilon$

$\rho_1(x_n, a) < 1/n$

Thus we get a seq. $\{x_n\}$ in $M_1 \ni$:

$\rho_2(f(x_n), f(a)) \geq \epsilon$, $\rho_1(x_n, a) < 1/n$

$\Rightarrow x_n \to a$ as $n \to \infty$.

But $[f(x_n)]$ does not converge to $f(a)$.

This is $\Rightarrow \Leftarrow$ $\therefore f$ must be conti at a.

Theorem: 7

Let (M_1, ρ_1), (M_2, ρ_2), (M_3, ρ_3) be metric spaces and let $f: M_1 \to M_2$, $g: M_2 \to M_3$. If f is conti at $a \in M_1$ and g is conti at $f(a) \in M_2$ then $g \circ f$ is conti at a.

Proof:

Let $h = g \circ f : M_1 \to M_3$ we have to S.T h is conti at a.

Let $f(a) = b$ and let $\epsilon > 0$ be given since g is conti at b $\exists$ a $\delta > 0$ $\ni$:

$\rho_3(g(y), g(b)) < \epsilon$ whenever $\rho_2(y, b) < \delta$ ------>(1)

But, then f is conti at a hence for this δ, there is a $\delta_1 > 0$ $\ni$: $\rho_2(f(x), f(a)) < \delta$

whenever $\rho_1(x, a) < \delta_1$ -------->(2)

Taking $y = f(x)$ we get,

$\rho_1(x, a) < \delta_1 \Rightarrow \rho_2(y, b) < \delta$

$\Rightarrow \rho_3(g(x), g(b)) < \epsilon$ by (1)

$\Rightarrow \rho_3(g(f(x)), g(f(a))) < \epsilon$ -------->(3)

(i.e) $h = g \circ f$ is conti at a.

Theorem: 8

Let M be a metric space and at f and g be real valued functions which are conti at $a \in M$. then $f+g$, $f-g$ and fg are also conti at a. Further if $g(a) \neq 0$ f/g is also conti at a.

Proof: Similar to those corresponding to conti fn in R^1 (Theorem (1))

Definition:

Let M_1 and M_2 be metric spaces and let $f: M_1 \to M_2$ we say that f is a conti fn. from M_1 into M_2 if f is conti at each point in M_1.

UNIT-V

CONTINUOUS FUNCTIONS ON METRIC SPACES

Theorem : 9

If f and g are continuous fn from a metric space M_1 into a metric space M_2 then so are $f+g$, $f-g$, fg and f/g where $g(x) \neq 0$, $x \in M_1$.

Examples

1. If f is continuous at a and if $c \in R$, prove that cf is continuous at a.

proof:

If $c=0$ then the theorem is nothing to prove. So assume $c \neq 0$.

Given $\epsilon > 0$ we must find $\delta > 0$ $\ni$:

$$|cf(x) - cf(a)| < \epsilon \quad \text{----------} \to (1)$$

whenever $|x-a| < \delta$.

Since f is continuous at a $\exists$ $\delta > 0$ $\ni$: for $|x-a| < \delta$

$$|f(x) - f(a)| < \frac{\epsilon}{|c|}$$

$\Rightarrow |c| \, |f(x) - f(a)| < \epsilon$ from this (1) follows.

Example : 2

If f is conti at a and $f(a) > 0$, prove that $\exists$ $h > 0$ $\ni$: $f(x) > 0$ $(a-h < x < a+h)$.

Soln: Let $\epsilon = \frac{f(a)}{2}$, $\exists$ $h > 0$ $\ni$: $|f(x) - f(a)| < \frac{f(a)}{2}$

whenever $|x-a| < h$

(i.e) whenever $-h < x-a < h$

(or) $a-h < x < a+h$

we have $-\frac{f(a)}{2} < f(x) - f(a) < \frac{f(a)}{2}$

(or) $\frac{f(a)}{2} < f(x) < 3/2\, f(a)$

Since $f(a) > 0$

$0 < \frac{f(a)}{2} < f(x)$

$\Rightarrow f(x) > 0$ for $a-h < x < a+h$

Example : 3

If f is conti at $a \in R'$ then $|f|$ is also conti at a (Thm (a))

Sol:- Let f be conti at a

Then given $\epsilon > 0$, $\exists$ $\delta > 0$ $\ni$:

$|f(x) - f(a)| < \epsilon$ --------> (1) whenever $|x-a| < \delta$.

But $||f(x)| - |f(a)|| \leq |f(x) - f(a)|$

$\therefore ||f(x)| - |f(a)|| < \epsilon$ using (1) $(|x-a| < \delta)$

$\therefore |f|$ is conti at a.

Example : 4

If $f(x) = x$, $(-\infty < x < \infty)$ P.T f is continuous at each point in R'.

Sol:- Let $a \in R'$ for $\epsilon > 0$ and let $\delta = \epsilon$.

Then for $|x-a| < \delta$, $|f(x) - f(a)| = |x-a| < \delta = \epsilon$

$\Rightarrow x$ is continuous at $a \in R'$

$\therefore a$ is arbitrary $f(x) = x$ is continuous at each point in R'.

Example: 5

If n is a +ve integer and $f(x)=x^n$ $(-\infty<x<\infty)$ then P.T f is conti at each point in R^1.

Soln: Let $f_1(x)=x$ for $(-\infty<x<\infty)$ then

W.K.T. f is continuous at each point x in R^1.

Now. $f(n)=(f_1 \cdot f_1 \cdot f_1 \cdots\cdots n\text{ times})x$

$= f_1^n(x)=x^n$

$\Rightarrow f_1^n$ is continuous

($\because$ product of conti fn is conti)

Hence $f(x)=x^n$ is conti in $(-\infty<x<\infty)$

5.4. Open sets

Definition:

Let M be a metric space. Let G be a Subject of M. We Say that G is an open subset of M (or G is open) if for every $x\in G$ $\exists$ a number $r>0$ $\ni$: the entire open ball $B[x,r]$ is contained in G.

Proposition:

Any open ball in a metric space (M,d) is an open set (OR).

Every open ball is an open set in M.

Proof: Let $B(a;r)$ be an open ball in M.

Let $x\in B(a,r)$ then $d(a,x)<r$.

put $\delta=r-d(a,x)$

Then $\delta>0$.

consider the open ball $B(x;\delta)$

Let $y\in B(x;\delta)$ then

$d(x,y) < \delta = r - d(a,x)$

$\Rightarrow d(a,x) + d(x,y) < r$ ...... $\rightarrow (1)$

Now, $d(y,a) = d(a,y)$

$\leq d(a,x) + d(x,y)$ by triangle inequality

$< r$ by (1)

$\because d(y,a) < r$. Hence $y \in B(a,r)$

$\therefore y \in B(x,\delta) \Rightarrow y \in B(a,r)$

$\because B(x,\delta) \subset B(a,r)$

Hence $B(a,r)$ is open set in M.

Remark:

(i) Consider R^1 with usual metric. If $a \in R^1$ then $\{a\}$ is not open in R^1. For every open ball in R^1 is a non-empty interval and certainly $\{a\}$ cannot contain such interval.

(ii) Consider $M = R_d$. If $a \in R_d$ then $\{a\} = B[a;1]$

Hence $\{a\}$ is an open set in R_d.

(i.e.) any set with only one point in it is open in R_d.

This shows that whether a set A is open or not, depends on what metric spaces is under consideration.

Bounded Sets:

A subset A of a metric space M is bounded if $\exists$ a +ve real numbers $k \ni$: $d(x,y) \leq k \ \forall x,y \in A$.

Example 1:

Every open ball is a bounded set.

Proof: Let $B(a;r)$ be an open ball with centre a and radius r.

Let $x, y \in B[a; r]$ then

$d(a, x) < r$, $d(a, y) < r$.

By triangle inequality,

$$d(x, y) \le d(x, a) + d(a, y)$$
$$= d(a, x) + d(a, y)$$
$$\le r + r = 2r$$
$$\therefore d(x, y) < 2r$$

Thus $B(a; r)$ is a bounded set.

Examples for open sets:

Consider the metric space $M = R^1$ with usual metric. Then

(i) Any open interval (a, b) is an open set

For let $x \in (a, b)$

Put $r = \min\{x - a, b - x\}$

Then $r > 0$ and $B(x, r) = (x - r, x + r) \subseteq (a, b)$

$\therefore (a, b)$ is an open set.

(ii) The interval $[1, 2]$ is not open. For there is no open ball with centre 1 contained in $[1, 2]$

(iii) The closed interval $A = [a, b]$ is not open ball with centre a or b contains points outside A.

(i.e.) there is no open ball with centre a or b contained in A.

Hence A is not open.

(iv) The infinite interval (a, ∞) and $(-\infty, a)$ are open sets where $a \in R^1$.

(v) Consider R^2 with usual metric.

Let $a = (a_1, a_2)$ where $a_1, a_2 \in R$.

$B[a, r] = \{x : (x_1, x_2) \in R^2 / \rho(x, y) < r\}$

$= \{x \in R^2 : \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < r\}$

$= \{x \in R^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2\}$

This is the open disk with centre at $a = (a_1, a_2)$ and radius r.

Theorem: 10

In any metric space (M, d) both M and the empty set ϕ are open sets.

Proof: If $x \in M$ then by defn of $B[x; r]$, every open ball $B[x; r]$ is contained in M.

Hence M is open.

ϕ is open since there are no x in ϕ and hence every $x \in \phi$ satisfies the condition in the definition for an open set.

Theorem: 11

If $\{G_i / i \in I\}$ is a family of open sets in a metric space M then $\bigcup_{i \in I} G_i$ is also an open subset of M.

Proof: Let $G = \bigcup_{i \in I} G_i$

If $G = \phi$ then by above theorem.

G is open.

Assume that $G \neq \phi$

Let $x \in G$ we have to prove that there is an open ball $B(x; r) \subset G$.

$\therefore x \in G = \bigcup_{i \in I} G_i$

$x \in G_i$ for some $i \in I$

But G_i is open.

$\therefore$ there is an open ball $B(x;r) \subset G$,

$B(x;r) \bigcup_{i \in I} G_i = G$

(ie) $B(x;r) \subset G$ and hence G is open.

Corollary: Every subset S of R_d is open.

soln: For if $a \in R_d$ then

$B(a;1) = \{x / d(a,x) < 1\} = \{a\}$

But every open ball is an open set.

thus all single point sets in R_d are open

Also any subset S of R_d is a union of single point sets. Hence by theorem 11, S is open.

Theorem: 12

If G_1 and G_2 are open subsets of a metric space M, then $G_1 \cap G_2$ is also open.

Proof: If $G_1 \cap G_2 = \phi$ then it is open.

$\therefore$ let $G_1 \cap G_2 \neq \phi$.

Let $x \in G_1 \cap G_2$ be arbitrary then

$x \in G_1$ and $x \in G_2$.

But G_1 and G_2 are open sets in M.

Hence there are +ve real numbers r_1 and r_2 such that,

$B(x;r_1) \subset G_1$ and

$B(x;r_2) \subset G_2$.

Let $r = \min(r_1, r_2)$ then $r > 0$ and

$B(x,r) \subset B(x,r_1) \subset G_1$

$B(x,r) \subset B(x,r_2) \subset G_2$.

$\therefore$ for each $x \in G_1 \cap G_2$ $\exists$ an open ball

$B(x,r) \ni B(x,r) \subset G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2$ is open.

Remarks:

By induction it follows that the intersection of any finite number open sets is also an open set.

the intersection of an infinite number of open sets in a metric space need not be open.

For example, in the metric space R^1 with usual metric.

Let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ $n = 1, 2, 3, \ldots\ldots$

Then I_n is open in R^1 for each n.

Also $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

which is not open in R^1.

Since any open ball with centre 0 is a non-empty open interval which is not contained in $\{0\}$

Theorem: 13

Every open subject G of R^1 be expressed as a union of countable number of mutually disjoint open intervals

soln: Let $x \in G$ since G is open $\exists$ an open ball $B(x, r) = (x-r, x+r) \subset G$.

Let $I = (x-r, x+r)$

thus there is an open interval I.

$\ni$: $x \in I_x$ and $I_x \subset G$

then $G = \bigcup_{x \in G} I_x$

Now if $x, y \in G$ then either

$I_x = I_y$ (or) $I_x \cap I_y = \phi$

Suppose $I_x \cap I_y \neq \phi$ we show that $I_x = I_y$

I_x and I_y are open intervals

$\ni: x \in I_x,\ I_x \subset G,$

$y \in I_y,\ I_y \subset G.$

Also $I_x \cap I_y \neq \phi$

$\Rightarrow I_x \cup I_y$ is an open interval.

$\because I_x \cup I_y$ is an open interval containing $x \ni: I_x \cup I_y \subset G.$

But I_x is the largest open interval containing $x: I_x \subset G.$

$\therefore I_x \cup I_y = I_x \Rightarrow I_y \subset I_x$

$\|^{ly}$ $I_x \subset I_y$

$\therefore I_x = I_y$

Let $S = \{ I_x / x \in G \}$

then S is a family of mutually disjoint intervals I_x.

For each I_x in S. choose a rational number q_x in I_x

Define a map $f: S \to Q$ by $f(I_x) = q_x$

then $I_x \neq I_y \Rightarrow q_x \neq q_y$

$\therefore f$ is 1-1

Thus S is equivalent to a subset of Q.

But Q is countable

$\therefore S$ is also countable.

Thus G is a union of countable number of mutually disjoint open intervals.

Theorem : 14

Let (M_1, ρ_1) and (M_2, ρ_2) be metric spaces and let $f: M_1 \to M_2$. then f is conti on M_1 iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2 (i.e f is conti iff the inverse image of every open set is open).

Proof: Let us suppose that f is conti on M.

We have to show that,

$f^{-1}(G)$ is open in M_1 whenever G is open in M_2

(i.e) if $x \in f^{-1}(G)$ we must find an open ball $B(x; r)$ contained in $f^{-1}(G)$.

Since $x \in f^{-1}(G) \Rightarrow f(x) = y \in G$.

As G is open in M_2 there is an open ball $B[y; \delta] \subset G$.

By theorem on conti fn.

$f^{-1}[B(y;\delta)]$ contains some $B(x; r)$

Hence $f^{-1}(G) \supset f^{-1}[B(y;\delta)] = B(x, r)$ which proves if part.

only if part:

Suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

We must show that f is conti on M_1

For this it is enough to show that f is conti at an arbitary point $a \in M_1$.

Let $B = B[f(a); \epsilon]$ be any ball about $f(a)$.

Then B is open in M_2 and so (by hypothesis) $f^{-1}(B)$ is open in M_1.

Since $a \in f^{-1}(B)$ and $f^{-1}(B)$ is open, ~~that~~ there is an open ball $B[a, \delta]$ contained in $f^{-1}(B)$ and

$\therefore$ By thm. f is contn at a.

5.5. closed sets.

Defn: Let E be a subset of the metric space M. A point $x \in M$ is called a limit point of E if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points in E which converges to x. The set $\bar{E}$ of all limit points of E is called the closure of E.

Note: Let $x \in E$. The sequence $\{x, x, \dots\dots\}$ converges to x since $\rho(x, x) = 0$. Thus if $x \in E$ then $x \in \bar{E}$. That is if E is any subset of the metric space M, then $E \subset \bar{E}$

Theorem: 15

Let (M, ρ) be a metric space and $E \subset M$. A point $x \in M$ is a limit point of E iff every open ball centred at x contains at least one point of E.

Proof:

Let x be a limit point of E and $B(x, r)$ be an open ball about x.

Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in E $\ni$: $\{x_n\}$ converges to x.

So $\exists$ n_0 $\ni$: $\rho(x_n, x) < r$ for $n \geq n_0$

(ie) $\rho(x_{n_0}, x) < R$

$\Rightarrow x_{n_0} \in B(x, r)$

$\therefore B[x;r]$ contains a point of E.

Conversely, suppose every open ball centred at x contains a point of E.

Then each open ball $B\left(x, \frac{1}{n}\right)$ contains a point x_n of E.

$\rho(x_n, x) < \frac{1}{n} \longrightarrow 0$ as $n \to \infty$

(i.e) $x_n \to x$ as $n \to \infty$.

So x is a limit point of E.

Defn:

Let E be a subset of the metric space M. We say that E is a closed subset of M if $E = \bar{E}$

Note: A set is closed iff it contains all its limit points.

Example:-

1. All singletons are closed.

$\therefore$ In any metric space no point other than x is a point of $\{x\}$ so $\{x\}$ contains all its limit points. (i.e) $\{x\}$ is closed.

Thus if $a \in R_d$ than the set $\{a\}$ is both open and closed.

2. Finite sets are closed in R with usual metric.

3. If $M = R$ with usual metric, $A = [0,1]$ is closed.

4. If $M = R$ with usual metric, $A = (0,1)$ 0 is a limit point of A since $\{\frac{1}{n}\}$ converges to 0. But $0 \notin A$, so A is not closed.

5. If $M = \mathbb{R}^2$ infinite lines are closed subsets of $\mathbb{R}^2$

6. If $M = \mathbb{R}^3$, planes are closed in $\mathbb{R}^3$.

Theorem 16:

If E is any subset of a metric space M then $\overline{E} = \overline{\overline{E}}$ That is $\overline{E}$ is a closed subset.

Proof:

Since $\overline{E} \subset \overline{\overline{E}}$ we have to prove $\overline{E} \supset \overline{\overline{E}}$

Let $x \in \overline{\overline{E}}$ be arbitrary

To show that $x \in \overline{E}$ it is enough to show that any open ball $B[x;r]$ contains a point of E.

Since $x \in \overline{\overline{E}}$ the Ball $B[x;r]$ contains a point $y \in \overline{E}$.

Let $s = \rho(x,y)$ and let t be any +ve number with $t < r - s$

Since $y \in \overline{E}$ the ball $B(y,t)$ contains a point $z \in E$.

But $\rho(x,y) = s$

$$\therefore \rho(x,z) \le \rho(x,y) + \rho(y,z)$$
$$< s + t$$
$$< s + r - s = r$$

Hence $z \in B(x;r)$

Thus $B(x;r)$ contains a point of E.

Hence the result

Theorem 17

In any metric space (M,ρ) the sets M and ϕ are both closed.

Proof:

Clearly M contains all its limit points and that ϕ has no limit points and hence contains all its limit points

Theorem: 18

If F_1 and F_2 are closed subsets of the metric space M, then $F_1 \cup F_2$ is also a closed set in M.

Proof:- Since F_1 and F_2 are closed.

$$F_1 = \overline{F_1} \text{ and } F_2 = \overline{F_2}$$

Let $x \in \overline{F_1 \cup F_2}$ then there is a seq. $\{x_n\}_{n=1}^{\infty}$ of points in $F_1 \cup F_2$ which converges to x.

But $\{x_n\}_{n=1}^{\infty}$ must have a subsequence consisting wholly at points in F_1 or a subsequence consisting of points in F_2.

Since any subsequence of $\{x_n\}_{n=1}^{\infty}$ must converge to x,

this implies that $x \in \overline{F_1} = F_1$ or $x \in \overline{F_2} = F_2$

thus $x \in F_1 \cup F_2$

$$\Rightarrow F_1 \cup F_2 \supset \overline{F_1 \cup F_2}$$

But $\overline{F_1 \cup F_2} \subset F_1 \cup F_2$

Hence $F_1 \cup F_2 = \overline{F_1 \cup F_2}$

i.e., $F_1 \cup F_2$ is closed.

Remark: By induction it follows that the union of any finite number of closed sets is also a closed set.

However the union of an infinite number of closed sets in a metric spaces need not be closed

For example, in the metric space R with usual metric.

Let $F_n = \left[\frac{1}{n}, 1\right]$, $n = 1, 2, 3, \ldots\ldots$

Then $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right]$

$= \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \ldots\ldots$

$= (0, 1]$ But $(0,1]$ is not closed in R

Hence $\bigcup_{n=1}^{\infty} F_n$ is not closed.

Theorem: 1.9

If $\mathcal{F}$ (script F) is any family of $\mathcal{F}$ closed subsets of a metric space M, then $\bigcap_{F \in \mathcal{F}} F$ is closed

Proof: Let $x \in \overline{\bigcap_{F \in \mathcal{F}} F}$

Then any Ball $B(x; r)$ contains a point $y \in \bigcap_{F \in \mathcal{F}} F$

$\Rightarrow y \in F$ for all $F \in \mathcal{F}$

Thus x lies in every $F \in \mathcal{F}$ and so

$x \in \bigcap_{F \in \mathcal{F}} F$

$\Rightarrow \bigcap_{F \in \mathcal{F}} F \supset \overline{\bigcap_{F \in \mathcal{F}} F}$

But we have

$\overline{\bigcap_{F \in \mathcal{F}} F} \subset \bigcap_{F \in \mathcal{F}} F \Rightarrow \bigcap_{F \in \mathcal{F}} F = \overline{\bigcap_{F \in \mathcal{F}} F}$

$\Rightarrow \bigcap_{F \in \mathcal{F}} F$ is closed.

Note:

The following theorem gives an important relation b/w open sets and closed sets namely, a set is open iff its complement is closed.

Theorem : 20

Let G be an open subset of the metric space M. Then $G' = M - G$ is closed. Conversely if F is closed subset of M, then $F' = M - F$ is open.

Proof: Let us first suppose that G is open.

If $x \in G$ then by definition of open set there is a ball $B = B(x, r)$ which lies entirely in G.

Hence B contains no points of G'.

By theorem,

x cannot be a limit point of G'.

thus no point in G is a limit point of G' and

So G' contains all its limit points.

Hence G' is closed.

Now, let us suppose that F is closed.

If $y \in F'$ there must be a ball $B(y, r)$ which contains no point of F.

For otherwise y would be a limit point of F which means $y \in F$ ($\because F$ is closed).

This contradicts $y \in F'$.

thus for every $y \in F'$ there is a ball $B(y, r)$ lying entirely in F'.

thus F' is open.

Note: (i) In any metric space both M and the empty set ϕ are closed.

Proof: We know by thm 10, both M and ϕ are open sets.

$M - M = \phi$ is open $\therefore M$ is closed.

$\Rightarrow M - \phi = M$ is open $\therefore \phi$ is closed.

(ii) If F_1 and F_2 are closed sets in a metric space M, then $F_1 \cup F_2$ is a closed set in M.

Soln: Let $F = F_1 \cup F_2$ then

$M - F = M - (F_1 \cup F_2)$

$= (M - F_1) \cap (M - F_2)$ by De-Morgan's law.

$\because F_1$ and F_2 are closed sets $M - F_1$ and $M - F_2$ are open

Also intersection of a finite number of open sets is open.

Hence $(M - F_1) \cap (M - F_2)$ is open.

(i.e) $M - F$ is open and so F is closed.

Theorem: 21

Let (M_1, ρ_1) and (M_2, ρ_2) be metric spaces and let $f: M_1 \to M_2$. Then f is continuous on M_1 iff $f^{-1}(F)$ is a closed subset of M_1 whenever F is a closed subset of M_2.

Proof: Let us suppose that f be continuous on M_1. If $F \subset M_2$ is a closed set, then by theorem $F' = M_2 - F$ is open.

Also by theorem $f^{-1}(F')$ is open in M_1.

Since $F \cup F' = M_2$ we have by a known result.

$f^{-1}(F) \cup f^{-1}(F') = f^{-1}(M_2)$

(i.e) $f^{-1}(F) \cup f^{-1}(F') = M_1$.

Hence $f^{-1}(F)$ is the complement (relative to M_1) of $f^{-1}(F')$.

Since $f^{-1}(F')$ is open then $f^{-1}(F)$ is closed.

Theorem 122

Let f be a 1-1 function from a metric space M_1 onto a metric space M_2. Then if f has ~~any~~ any one of the following properties it has them all.

(a) Both f and f^{-1} are continuous (on M_1 and M_2 respectively).

(b) The set $G \subset M_1$ is open iff its image $f(G) \subset M_2$ is open.

(c) The set $F \subset M_1$ is closed iff its image $f(F)$ is closed.

Proof:

We shall prove (a) $\Rightarrow$ (b)

Let us assume that both f and f^{-1} are continuous.

Since f is 1-1 and onto.

f^{-1} is well-defined.

Let G be open in M_1.

Since f^{-1} is continuous $(f^{-1})^{-1}(G) = f(G)$ is open in M_2.

Conversely, let $f(G)$ be open in M_2 as f is continuous, $f^{-1}(f(G)) = G$ is open in M_1.

Hence G is open in M_1.

To Prove (b) $\Rightarrow$ (c)

Let F be a closed set in M_1

$\therefore$ By a Theorem $M_1 - F$ is open in M_1.

by (b) $M_1 - F$ is open in $M_1 \Leftrightarrow f(M_1 - F)$ is open in M_2

$\Leftrightarrow f(M_1 - F) = f(M_1) - f(F)$

$= M_2 - f(F)$ is open in M_2.

$\Leftrightarrow f(F)$ is closed in M_2.

To Prove (c) $\rightarrow$ (a)

Let F be closed in M_1.

$\Rightarrow (f^{-1})^{-1}(F) = f(F)$ is closed in M_2

Hence f^{-1} is continuous.

Let us assume that,

$f(F)$ is closed in M_2,

then F is closed by (c)

But $F = f^{-1}(f(F))$

Hence for $f(F)$ closed in M_2,

$f^{-1}(f(F)) = F$ is closed in M_1

Hence f is continuous.

Homeomorphism:

A one to one and onto function f which is also continuous defined from a metric space M_1 to M_2 is called a homeomorphism.

Remark: If f has any one (and hence all) of the properties mentioned in the above theorem we call f a homeomorphism from M_1 onto M_2.

If a there exists a homeomorphism from M_1 onto M_2, we say that M_1 and M_2 are homeomorphic.

Example:

let $f: [0,1] \to [0,2]$ be defined by

$f(x) = 2x$.

The metric for $[0,1]$ and $[0,2]$ are absolute value metric.

Then f is a homeomorphism of $[0,1]$ onto $[0,2]$

III^{ly} if $f(x) = \log x$ then f is a homeomorphism of $(0,\infty)$ onto R^1.