13.6.19

Unit - I.

Topological Spaces

Def: (Topological) Topology

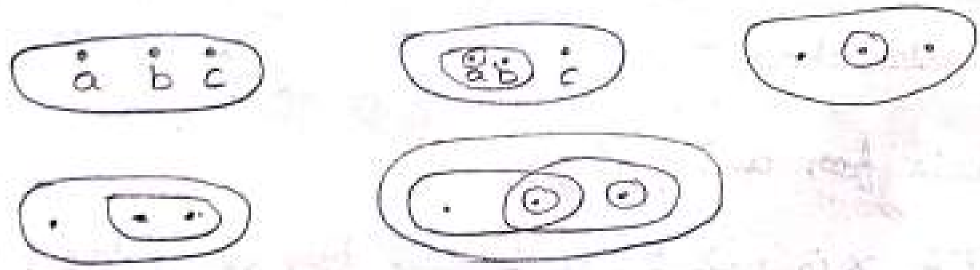
A topology on a set x is a Collection τ of Subsets x having the following.

Properties:

1) ϕ & x are in τ
2) The Union of the elements of any Subcollection τ is in τ
3) The intersection of the elements of any finite Subcollection of τ is in τ.

A set 'x' for which a topology τ has been specified is called a topological Space.

Ex: Let x be a 3 element set $x = \{a, b, c\}$ here are the variableous topologis listed under in diagram-matic form:-



Ex: 2

If x is any set. the collection of all subset of x is a topology on x; it is called the discrete topology.

The collection Consisting of x and ϕ only is also a topology on x called indiscrete topology (or) trivial topology.

14.6.19

Defn:.

Suppose τ and τ' are two topologies on a given set X if $\tau' \supset \tau$, then τ' is finer than τ. If τ' Property contains τ, then τ' is strictly finer than τ. We also say that τ is coarser than τ' (or) strictly coarser in these above situation

We say τ is comparable with τ' if either τ' is $\tau' \supset \tau$ (or) $\tau \supset \tau'$. In this situation either τ' is larger than τ (or) τ is ~~larger~~ Smaller than τ', Equivalently for second inclusion. we say τ is larger than τ' (or) τ' is smaller than τ.

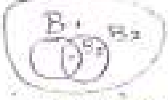
5(K)
6m (1) (iii)

Defn: Basis for a Topology.

If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) Such that

i) For each $x \in X$, there is atleast one basis element B containing x.

ii) If $x \in B_1 \cap B_2$, then there is a basis element B_3 containing x such that $B \subset B_1 \cap B_2$



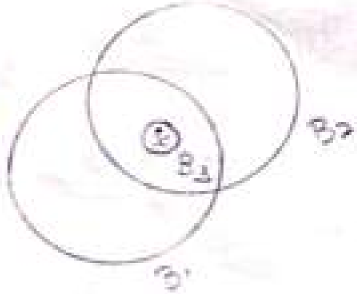
If $\mathcal{B}$ satisfies these two conditions, then define the topology τ generated by $\mathcal{B}$ as follows.

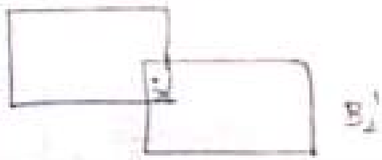
A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ $\ni: x \in B$ and $B \subset U$

Note:- Each basis element is itself an element of τ.

18.6.19 Example:

Let $\mathcal{B}$ be the collection of all circular regions (interior of circles) in the plane then $\mathcal{B}$ satisfies both conditions for a basis.





Lemma 1.1 (2) I-term

Let X be a set. Let $\mathcal{B}$ be a basis for the topology τ on X. Then τ equals the collection of all union of elements of $\mathcal{B}$.

Proof:- Let X be a set. Let $\mathcal{B}$ be a basis for a topology τ on X. Clearly members in $\mathcal{B}$ are also members in τ.

As τ is a topology, their union is also in τ.

$B \in \mathcal{B} \Rightarrow \tau$

$B_1 \cup B_2 \cup B_3 \in \tau$

Hence τ equals is the family of all union of basic open sets

Given $U \in \tau$. For each $x \in U$, choose a basis element $B_x \in \mathcal{B}$

$\ni \; x \in B_x \subset U$

Hence we write $U = \bigcup_{x \in U} B_x$. This shows that the open set U in τ equals a union of members of $\mathcal{B}$

S(2)

Imp Lemma 1.2: Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets in X $\ni$ for each $x \in X$ and each open set $U \in \tau$, there is a member $C \in \mathcal{C}$ with $x \in C \subset U$ then $\mathcal{C}$ is a basis for the topology τ on X.

1.6.19

Proof:-

To show that $\mathcal{C}$ is a basis

I - Term (3)

Condition (i): Given that $x \in X$.

$\because$ X itself is open set, by hypothesis $\exists$ an element $C \in \mathcal{C}$, $\ni$ $x \in C \subset X$.

Condition (ii) Let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$

$\because$ C_1, C_2 are open

$\Rightarrow C_1 \cap C_2$ is also open

Then by defn of open set C_3,

$\ni: x \in C_3 \subset C_1 \cap C_2$.

$\therefore$ From (i) & (ii) $\mathcal{C}$ is a basis for X.

To show that the topology τ' generated by $\mathcal{C}$ equals the topology τ. If $U \in \tau$ and if $x \in U$, Then by hypothesis $\exists$ an element $C \in \mathcal{C}$, $\ni$ $x \in C \subset U$. $\Rightarrow U \in \tau'$

$\therefore U \in \tau \Rightarrow U \in \tau'$

$\therefore \tau \subset \tau'$

Conversely if $W \in \tau'$

Then w equals the union of elements of $\mathcal{C}$

(i.e) $W = \bigcup_{x \in X} C_x$.

$\because$ each $C_x \in \tau$ As $C_x \subset U$.

$\Rightarrow W \in \tau$.

$\therefore W \in \tau' \Rightarrow W \in \tau$

(i.e) $\tau' \subset \tau$ ——— (2)

From (1) & (2) $\tau = \tau'$.

(18) 15m (i) S

15m T-I

Lemma : 1.3
Let $\mathcal{B}$ & $\mathcal{B}'$ be bases for the topologies τ and τ' respectively on X. Then the following are equivalant

i) τ' is finer than τ.

ii) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, There is a basis element $B' \in \mathcal{B}'$

21.6.19 $\ni: x \in B' \subset B$.

15m (1) I. Term

Proof :-

To prove (2) $\Rightarrow$ (1)

Given an element U of τ. Let $x \in U$.

Given that $\mathcal{B}$ is a basis for the topology τ on X.

$\Rightarrow$ $\mathcal{B}$ generates τ.

$\Rightarrow$ By defn. of basis

$\exists$ $x \in B \subset U$.

By Assumption (2) $x \in B' \subset B$.

But $B \subset U$.

$\Rightarrow B' \subset U$ and $U \in \tau'$

$U \in \tau \Rightarrow U \in \tau'$

$\therefore \tau \subset \tau'$ (or $\tau' \supset \tau$.

$\Rightarrow \tau'$ is finer than τ.

To prove (ii) ⇒ (i)

Given that $x \in X$ and $B \in \mathcal{B}$ with $x \in B$.

($\mathcal{B}$ is basis that generates

$\Rightarrow B \in \mathcal{T}$

by (i) $\mathcal{T}'$ is finer than $\mathcal{T}$

$\Rightarrow \mathcal{T}' \supset \mathcal{T}$

(or) $\mathcal{T} \subset \mathcal{T}'$

$B \in \mathcal{T} \Rightarrow B \in \mathcal{T}'$ (basis condition $x \in B \subset U$)

$\mathcal{T}'$ is generated by $\mathcal{B}'$

$\Rightarrow$ by defn of basis $\exists$ a basis element $x \in B' \subset B$

Defn : Standard Topology. ($\mathcal{B}$ (or) $\mathcal{B}$ → basis.

If $\mathcal{B}$ is the collection of all open intervals in the real line.

$(a,b) = \{x \mid a < x < b\}$.

The topology generated by $\mathcal{B}$ is called the Standard topology on the real line.

Defn : Lower Limit Topology.

If $\mathcal{B}'$ is the collection of all half eqn intervals of the form

$[a,b) = \{x \mid a \le x < b\}$ where $a < b$,

the topology generated by $\mathcal{B}'$ is called the lower limit topology denoted by $\mathbb{R}_l$

Defn : K - topology

Let K - denote the set of all number of the form $\frac{1}{n}$ for $n \in \mathbb{Z}^+$ and let $\mathcal{B}''$ be the collection of open intervals (a,b) along with all the sets form: $(a,b) - K$.

The topology generated by $\mathcal{B}''$ will be called the k-topology. This is denoted by $\mathbb{R}_K$.

Lemma 1.4:

25.6.19

The topologies of $\mathbb{R}_l$, and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$

Proof:-

Let τ, τ' and τ'' be the topologies of $\mathbb{R}$, $\mathbb{R}_l$ and $\mathbb{R}$ respectively.

Given a basis element (a,b) for τ and a point x of $(a,b]$, the basis element $[x,b)$ for τ' contains x and lies in (a,b).

Given the basis element $[x,d)$ a for τ', there is no open (a,b) that contains x and lies in $[x,d)$.

$\therefore$ τ' is strictly is finer than τ

$|||^{rly}$ τ'' is strictly finer than τ

Def: Sub basis

A Subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection τ of all union of finite intersection of elements of $\mathcal{S}$.

Def: Order topology. (6m) S 17 (ii)

Let X be a set with a Simple order relation Assume that X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types.

i) All open intervals (a,b) in X

ii) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

iii) All intervals of the form $[a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X which is called the order topology.

Example :

The Standard topology on $\mathbb{R}$ is the order topology from the usual order in $\mathbb{R}$

Defn : Rays .

If X is an ordered set and a is an element of X, there are four subsets of X that are called rays determined by a. They are as follows..

$$(a, \infty) = \{x / x > a\}$$

$$(-\infty, a) = \{x / x < a\}$$

$$[a, \infty) = \{x / x \geq a\}$$

$$(-\infty, a] = \{x / x \leq a\}$$

Sets of these first two types are called operation rays. Sets of last two types are called closed rays.

26.6.19.

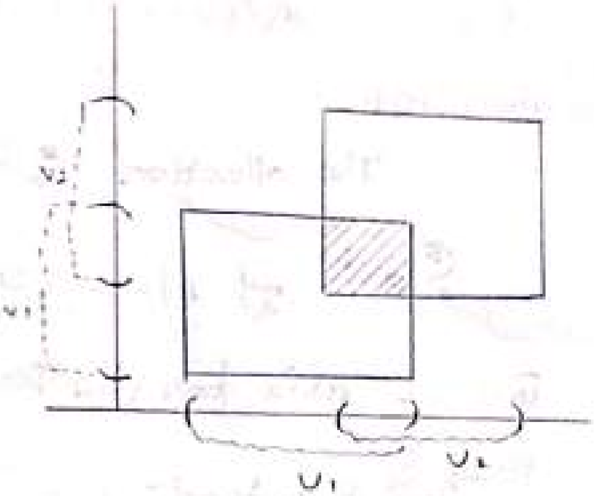
Product Topology

Let X and Y be topological spaces. The product topology $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is open subset of X and V is open subset of Y.

Theorem :- 1.1 (5m) 2 T-term

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection $\mathcal{D} = \{B \times C \,/\, B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$

Proof:-



Given an open set W of $X \times Y$ and a point $x \times y$ of W.

by defn of Product topology, there is a basis element $U \times V$

$\ni: x \times y \in U \times V$,

$\ni \; x \times y \in U \times V \subset W$.

As $\mathcal{B}$ and $\mathcal{C}$ are bases for X & Y respectively, choose $x \in B \subset U$ and $y \in C \subset V$. Also $x \times y \in B \times C \subset W$.

by a lemma (1.2)

the collection $\mathcal{D}$ is a basis for the topology of $X \times Y$

Example:

(for product topology) The Product of the standard topology on $\mathbb{R}$ with itself (i.e) $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is a topology called the product topology.

The collection of all Products $(a,b) \times (c,d)$ of open intervals in $\mathbb{R}$ will serve as a basis for the topology of $\mathbb{R}^2$.

Defn:

28.6.19

Let $\pi_1 : X \times Y \to X$ be defined by

$\pi_1(x,y) = x$.

Let $\pi_2 : X \times Y \to Y$ be defined by

$\pi_2(x,y) = y$.

the maps π_1 & π_2 are called the projections of $X \times Y$ onto it's first and second factors respectively.

If U is an open subset of X, then the set $\pi_1^{-1}(U)$ is precisely the set $U \times Y$

$U \times Y = \{x \times y / x \in U$

$X \times V = \{x \times y / y$

$|||^{ly}$ if V is an open subset of Y the $\pi_2^{-1}(V) = X \times V$ open in $X \times Y$.

Theorem 1.2: J-term

The collection $\mathcal{S} = \{$ (3) (6m)

$$\mathcal{S} = \{\pi_1^{-1}(U) / U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) / V \text{ is open in } Y\}$$

is a sub basis for the Product topology on $X \times Y$.

Proof:-

Let τ denote the product topology on $X \times Y$.

Let τ' be the topology generated by $\mathcal{S}$.

Any element of $\mathcal{S}$ is also an element of τ.

$\Rightarrow$ Arbitrary Union of finite intersection of element of τ & $\mathcal{S}$ is also in τ.

$\therefore$ The element of τ are also element of τ.

$\therefore \tau' \subset \tau$ — ①

$\{\pi_1^{-1}(U_1) \cap \pi_1^{-1}(U_2) \dots \cap \pi_1^{-1}(U_n)\}$

$\cup \{\pi_2^{-1}(V_1) \cap \pi_2^{-1}(V_2) \dots \cap \pi_2^{-1}(V_n)\}$

On the other hand, every basis element $U \times V \in X \times Y$ for the topology τ is a finite intersection of element of $\mathcal{S}$

$\Rightarrow U \times V \in \mathcal{S}$

ie $U \times V \in \tau'$ $\because U \times V \in \tau \Rightarrow U \times V \in \tau'$

$\tau \subset \tau'$ — ②

from ① & ②

$\tau = \tau$

$\mathcal{S}$ is a basis for τ'

$\Rightarrow \mathcal{S}$ is a basis for τ. But $\mathcal{B}$ is a basis for τ.

$\Rightarrow \mathcal{S}$ is a subbasis for τ on $X \times Y$.

3.7.'19

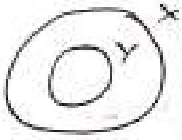
Defn: Subspace Topology. 6m (17) S (iii)

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection $\mathcal{T}_Y = \{Y \cap U / U \in \mathcal{T}\}$ is a topology on Y, called a subspace topology.

With this topology, Y is called a subspace of X.

(4) I-Term 6m

To check that $\mathcal{T}_Y$ is a topology:



1) $\phi = \phi \cap Y$ and $Y = Y \cap X$ are in $\mathcal{T}_Y$

2) $(U_1 \cap Y) \cap (U_2 \cap Y) \cap (U_3 \cap Y) \cap \dots \cap (U_n \cap Y)$

$= (U_1 \cap U_2 \cap U_3 \dots \cap U_n) \cap Y$.

$= U \cap Y \in \mathcal{T}$, where $U = U_1 \cap U_2 \cap \dots \cap U_n$

finite intersection of any subcollection of $\mathcal{T}_Y$ is in $\mathcal{T}_Y$.

3) $\bigcup_{\alpha \in I} U_\alpha \cap Y = (U_1 \cap Y) \cup (U_2 \cap Y) \dots (U_\alpha \cap Y)$.

$= (U_1 \cup U_2 \cup U_3 \dots \cup U_\alpha) \cap Y$

$= U \cap Y \in \mathcal{T}_Y$ where $U = U_1 \cup U_2 \cup \dots U_\alpha$.

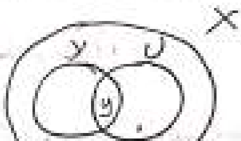
$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \cap Y \in \mathcal{T}_Y$

$\therefore$ Arbitary union of any subcollection of $\mathcal{T}_Y$ is in $\mathcal{T}_Y$

From (1) (2) & (3) $\mathcal{T}_Y$ is a topology for the subspace Y.

Lemma 1.5: (7) I-Term

(6m) If $\mathcal{B}$ is a basis for the topology of X, then the collection $\mathcal{B}_Y = \{B \cap Y / B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.



Proof:

Take an open set U in X and $y \in U \cap Y$.

As $\mathcal{B}$ is a basic for the topology $\mathcal{T}$ on X, by defn. of basis $\exists$ a basis element $B \in \mathcal{B}$, $\ni x \in B \subset U$

$y \in B$, $y \in Y \Rightarrow y \in B \cap Y$.

But $B \subset U$.

$\Rightarrow B \cap Y \subset U \cap Y$

$\therefore y \in B \cap Y \subset U \cap Y$.

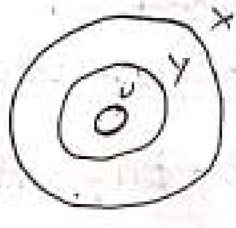
by lemma 1.2 (by a lemma)

$\mathcal{B}_Y$ is a basis for Y.

Lemma 1.6

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X

Proof:-



Let U be an open set in Y.

$\Rightarrow U = Y \cap V$ for some open set V in X.

$Y \cap V$ is open in X as Y & V are open in X.

(i.e) U is open in X.

6m (S) (tp) (X)

Theorem 1.3: 6m ✓ I-T

If A is a subspace of X and B is a subspace of Y, then the Product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof:-

The set $U \times V$ is the general basis element of $X \times Y$.

Now $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$

$\therefore$ $U \cap A$ & $V \cap B$ as general open sets for the subspace topologies on A & B.

The set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.

The basis element for the subspace topology on $A \times B$ and the Product topology on $A \times B$ are the same.

5.7.19

Closed sets and Limit Points.

Defn: closed set:

A subset A of a topological space X is said to be closed if the X − A is open.

Example: The subset [a, b] of $\mathbb{R}$ is closed since it's complement $\mathbb{R}$ [a,b] is open

$\mathbb{R} - [a,b] = (-\infty, a) \cup (b, \infty)$

$(-\infty, a)$ & (b, ∞) are open.

$\mathbb{R} - [a-b]$ is open.

(i.e) [a,b] is closed.

Theorem: 1.5

Let X be a topological space. There the following conditions hold:

1) ϕ and X are closed. Imp 15m (3)

2) Arbitrary intersection of closed sets are closed.

3) Finite union of closed sets are closed.

Proof:- To prove (1):

As ϕ and X are complement to each other

$\Rightarrow$ ϕ and X are closed.

To prove (2):

Given a collection of closed sets. $\{A_\alpha\}_{\alpha \in I}$

by demorgan's law.

$$\left(\bigcap_{i=1}^{n} A_i \right)' = \bigcup_{i=1}^{n} A_i'$$

$$X - \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X - A_\alpha)$$

Here each A is closed

$\therefore \cup (X - A_\alpha)$ is open $\Rightarrow X - A_\alpha$ is open for each X.

(i.e) $X - \bigcap_{\alpha \in I} A_\alpha$ is open

$\Rightarrow \bigcap_{\alpha \in I} A_\alpha$ is closed.

To prove (3):

Given a finite collection A_i, $i = 1, 2, \ldots n$ are closed for each i.

By demorgan's law

$$\left(\bigcup_{i=1}^{n} A_i \right)' = \bigcap_{i=1}^{n} A_i'$$

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$$

As each A_i is closed

$\Rightarrow X - A_i$ is open for each i

$\bigcap_{i=1}^{n} (X - A_i)$ is open

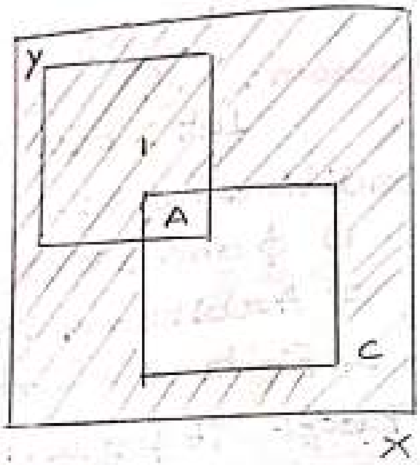
ie) $X - \bigcup_{i=1}^{n} A_i$ is open $\Rightarrow \bigcup_{i=1}^{n} A_i$ is closed

1 6m

(1) (P)

Theorem: 1.6 I.T 6m Let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y.

S (18) S (17) S (15) S 6m 13 3-time



proof: Direct part:

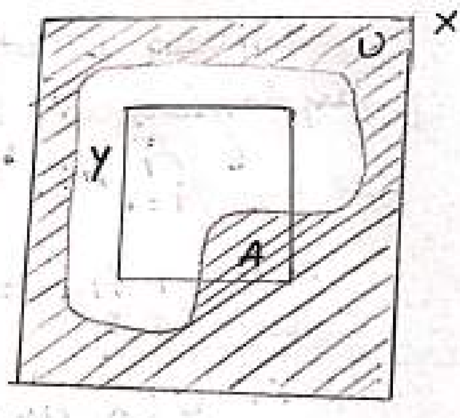
Assume that $A = C \cap Y$ where C is closed in X.

claim: A is closed in Y

C is closed in $X \Rightarrow X - C$ is open in X.

$\Rightarrow (X - C) \cap Y$ is open in Y

Converse part:



Assume that A is closed in Y.

$\Rightarrow Y - A$ is open in Y by defn. of closed set.

From diagram $A = Y \cap (X - U)$

where U is open in X

$\Rightarrow X - U$ is closed in X.

9.7.19

Closure and Interior of a set.

Given a subset A of a topological space X, then the interior of A is defined as the union of all open sets contained in A. Interior of A is denoted by

Int. A.

The charac' Closure of A is defined as the intersection of all closed sets containing A. Closure of A is denoted by Cl.A (or) $\overline{A}$)

Also Int. A is an open set and Cl.A is a closed set.

$Int\ A \subset A \subset \overline{A}$

If A is open, then $A = Int.A$

If A is closed, then $A = \overline{A}$.

Theorem 1.7.

Let Y be a subspace of X. Let A be a subset of Y. Let $\overline{A}$ denote the closure of A in X. then the closure of A in Y equals $\overline{A} \cap Y$. (6) I-term

Let $B = \overline{A}$ in Y

Given that $\overline{A}$ is closed in X.

$\Rightarrow A \subset \overline{A}$

$A \cap Y \subset \overline{A} \cap Y$

Also given that $\overline{A}$ is closed in X.

$\Rightarrow$ by a thm (1.6), $\overline{A} = \overline{A} \cap Y$ { $\overline{A} - \overline{A} \cap Y$
$\overline{A} \cap Y - \overline{A}$ }

$A \subset \overline{A} \Rightarrow A \subset \overline{A} \cap Y$

by def. of Closure of A,

B equals the intersection of all closed sets of Y containing A.

$\Rightarrow B \subset \overline{A} \cap Y$ — (1)

Again by a thm (1.6)

$B = C \cap Y$, where C is closed in X, Containing A

As $\overline{A}$ is closure of A, $A = \overline{A} \subset C$.

$\overline{A} \cap Y \subset C \cap Y$.

(i.e) $\overline{A} \cap Y \subset B$ — (2)

From (1) & (2) $B = \overline{A} \cap Y$.

(PTO)

Theorem 1.8 (15m) 5(ii) (B)

(5m) (A) Let A be a subset of the topological space X. Then

a) $x \in \bar{A}$ iff every open set U containing x intersect A.

b) Suppose the topology of X is given by on basis. Then

10.7.19 $x \in \bar{A}$ iff every basis element B containing x intersect A.

Proof:-

To prove (a) (proof by Negation)

Direct part:

$x \in \bar{A} \Rightarrow \exists$ an open set U containing x that does not intersect A.

Let $x \in \bar{A} \Rightarrow \exists$ an open set $U = X - \bar{A}$ containing x that does not intersect A.

Converse part:

If $\exists$ an open set U containing x that does not intersect A, Then a closed set $X - U$ containing A.

$\Rightarrow$ By defn of closure of A (or) $\bar{A}$

$X - U$ must contain A

$X - U$ must contain $\bar{A}$

By defn of complement of set

$x \in U \Rightarrow x \in X - U$

$\Rightarrow x \notin A$.

To prove (b)

Direct part:

Let $B \in \mathcal{B} \Rightarrow$ by defn of basis

$\exists$ a basis element $B \in \mathcal{B}$,

$\Rightarrow x \in B \subset U$.

where U is open in A

$\Rightarrow$ B containing x intersect A.

Converse part :

If every basis element $B \in \mathcal{B}$.

Containing x intersect A

$\Rightarrow$ Every open set U containing x interasect A

$\Rightarrow x \in A$

15m (i) (14) $\Rightarrow x \in \overline{A}$.

Defn: Limit Point :

If A is a subset of the topological space X and if x is a point X, Then x is a limit point (or claster point or point of accumulation) of A if every neighbourhood of x intersect A in Some Point other then x itself.

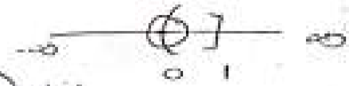
(or)

x is a limit point of A if it belongs to the closure $\underline{\text{of}}$ $A - \{x\}$

$A - \{x\}$

Example :

Consider the real line $\mathbb{R}$. If $A = [0,1]$ then the point 0 is a limit point of A.



15m (ii) (16) 15m (ii)

Theorem 1.9.

Let A be a Subset of the topological Space X. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$

Proof: If $x \in A'$ $\Rightarrow x$ is a limit point of A

$\Rightarrow x \in A \cup A'$

$\Rightarrow$ by defn of limit every nbd of x intersect A in Some point other than x.

$\Rightarrow$ by a thrm $\{1 - f(a)\}$, $x \in \overline{A}$

$x \in A \cup A' \Rightarrow x \in \overline{A}$

(i.e) $A \cup A' \subset \overline{A}$

(-∞) also

Let $x \in \bar{A} \Rightarrow x$ may lie in A (or) not

If $x \in A \Rightarrow x \in A \cup A'$.

$x \in A' \Rightarrow x \in A \cup A'$

$\therefore \bar{A} \subset A \cup A'$. — (1)

Suppose $x \notin A$

$\therefore x \in \bar{A} \Rightarrow$ by a Thm (1.8)

Every nbh of x intersects A in some there than x

$\Rightarrow x$ is a limit point of A

$\Rightarrow x \in A'$.

$\Rightarrow x \in A \cup A'$

$\therefore x \in A' \Rightarrow x \in A \cup A'$.

$\bar{A} \subset A \cup \bar{A}$ — (2)

from (1) & (2)

$\bar{A} \cup A \cup A'$.

Proof :-

Direct part :

Assume that every nbd of x.

contains infinitely many points of A.

$\Rightarrow$ The nbd of x intersect A in some point other than x.

$\Rightarrow$ by defn of limit point, x is the limit point of A.

Converse part :

Suppose x is a limit point of A.

Assume that some nbd U of x intersects A in only finitely many points.

$\Rightarrow$ U also intersects $A - \{x\}$ in finitely many pts.

If $\{x_1, x_2 \ldots x_m\}$ be the points of $U \cap (A - \{x\})$

Now the set $X - \{x_1, x_2 \ldots x_m\}$ is an open set.

$\therefore$ $U \cap \{X - \{x_1, x_2 \ldots x_m\}\}$ is a nbd of x that intersect $A - \{x\}$ not all

which is a $\Rightarrow\Leftarrow$ to assumption.

As $x_1, x_2 \ldots x_m \in U$ intersect $A - \{x\}$
$\Rightarrow \{x_1, x_2 \ldots x_m\} \in \bar{A}$
$\Rightarrow \{x_1, x_2 \ldots x_m\}$ is closed

$\therefore$ Every nbd of x interesect initely many points of A.

Theorem : 1.10

Every finite point set in a housdroff space X is closed.

Proof :- To prove this theorem it is enough to show that every singleton set $\{x_0\}$ is closed.

If x is point of X different from x_0 $\overset{(i.e)}{(x \neq x_0)}$ then the corresponding nbd's of x & x_0 are also distinct.

Now $\because$ U does not intersect $\{x_0\}$, the point cannot belong to the closure of the set $\{x_0\}$

$\Rightarrow$ The closure of the set $\{x_0\}$ is itself $\overline{\{x_0\}} = \{x_0\}$

$\Rightarrow \{x_0\}$ is closed.

This is true for every singleton set in X.

$\therefore$ Every finite point set in x is closed.

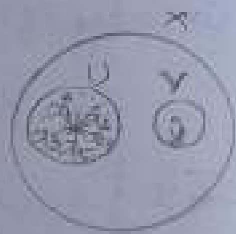
18.7.19

Theorem 1.12

If X is a Hausdorff Space then a Sequence Points of X Converges to atmost one point of X.

Proof:- Suppose that x_n is a Sequence of Points of X That Converge to x. If $y \neq x$.

Let U & V to be disjoint nbd's of x & y respectively. Since U contains x_n $\forall$ n but finitely many values of n, the Set V con't



$\therefore$ x_n con't Converge to y

$\therefore$ x_n converge to one Point (x) only.

Unit - II

Continuous Functions

Continuity of a function

Let X and Y be topological space a function $f: X \to Y$ is said to be continuous if ~~fo~~ for each open subset V of Y. the set $f^{-1}(V)$ is an open subset of X.

Example 1

Let us consider a function like the studied in analysis a real valued function of a real variable

$$f: \mathbb{R} \to \mathbb{R}.$$

Example 2:

In calculus one consider the property of continuity for many kinds of fn. for example, one studies fns. of the following types.

$f: \mathbb{R} \to \mathbb{R}^2$ (curves in the plane).

$f: \mathbb{R} \to \mathbb{R}^3$ (curves in space)

$f: \mathbb{R}^2 \to \mathbb{R}$ (fns. $f(x,y)$ of two real variables

$f: \mathbb{R}^3 \to \mathbb{R}$ (fns. $f(x,y,z)$ of three real variables

$f: \mathbb{R}^2 \to \mathbb{R}^2$ (vector field $v(x,y)$ the plane)

Each of them has a notion continuity defined for it. our general definition of continuity includes all these as special cases. This fact will be a consequence of general theorems are shall prove concerning continuous fns. on product spaces and on met metric spaces.

Example: 3

Let $\mathbb{R}$ denote the set of real numbers in its usual topology, and let $\mathbb{R}_l$ denote the same set in the lower limit topology. Let

$$f : \mathbb{R} \to \mathbb{R}_l$$

be the identity fun. $f(x) = x$ for every real number x. Then f is not continuous fun. the inverse image of the open set $[a, b)$ of the equals itself, which is not open in the $\mathbb{R}$ on the other hand the identity fun.

$$g : \mathbb{R}_l \to \mathbb{R}.$$

is continuous because the inverse image of (a, b) is itself which is open in $\mathbb{R}_l$.

1

2

10m 5 16

Theorem 2.1

3m 8

5m

Let X and Y be topological spaces, let $f : X \to Y$ Then the following are equivalent.

i) f is continuous.

ii) For every subset A of X, $f(\bar{A}) \subset \overline{f(A)}$

iii) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

iv) For each $x \in X$ and each neighbourhood V of $f(x)$ there is a neighbourhood U of x $\ni : f(U) \subset V$.

Proof:-

To Prove (i) $\Rightarrow$ (ii)

Suppose $f : X \to Y$ be continuous. Let $A \subset X$.

To prove: $f(\bar{A}) \subset \overline{f(A)}$ choose any arbitary pt $x \in \bar{A}$

$$\Rightarrow f(x) \in f(\bar{A}).$$

Let V be any neighbourhood of $f(x) \Rightarrow f^{-1}(V)$ (f is continuous)

is open set in X containing x and also it intersects A in some point y other than x.

Then V intersects $f(A)$ in some point $f(y)$

$\Rightarrow f(x) \in \overline{f(A)}$

As $f(x)$ is arbitary ~~continuous~~ element in $\overline{f(A)}$

$\Rightarrow f(\overline{A}) \subset \overline{f(A)}$

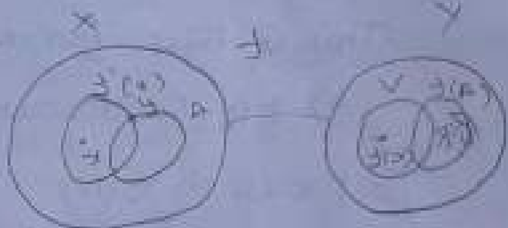
To prove (ii) $\Rightarrow$ (iii)

Let $f: X \to Y$ be onto

So that $y = f(x)$ and $f^{-1} = y \to x$.

Let B be closed set in Y. Take $A = f^{-1}(B)$ then is a neighbourhood V of x.

$\ni: f(U) \subset V$.



Claim:

A is closed in X.

(i.e) To show that $\overline{A} \subset A$

From the set them the

$f(A) \subset B$

$A = \overline{A}$
$A \subset \overline{A}$
$\overline{A} \subset A$

If $x \in \overline{A}$ then $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subset \overline{B} = B$

$\Rightarrow f(x) \in B$

(i.e) $x \in f^{-1}(B)$

$x \in A$

$f(A) \subset B$
$\overline{f(A)} \subset \overline{B}$
B is closed any
$\Rightarrow B = \overline{B}$

$x \in \overline{A} \Rightarrow x \in A$, $A \subset \overline{A}$

Illy

$\overline{A} \subset A \Rightarrow A$ is closed in X, $A = \overline{A}$

(i.e) $f^{-1}(B)$ is closed in X.

To prove (iii) $\Rightarrow$ (i)

Assume $f^{-1}(B)$ is closed in X for every closed set B in Y.

Claim :-

f is Continuous

Choose an arbitary open set V in Y so that $B = Y - V$ is closed in X.

by assumption $f^{-1}(B)$ is closed in X.

$$V = Y - B$$

$$f^{-1}(V) = f^{-1}(Y - B)$$

$$= f^{-1}(Y) - f^{-1}(B)$$

$$= X - f^{-1}(B)$$

$X - f^{-1}(B)$ is open in X. $f^{-1}(V)$ is open in X.

V is open in Y.

$\Rightarrow f^{-1}(B)$ is closed in X.

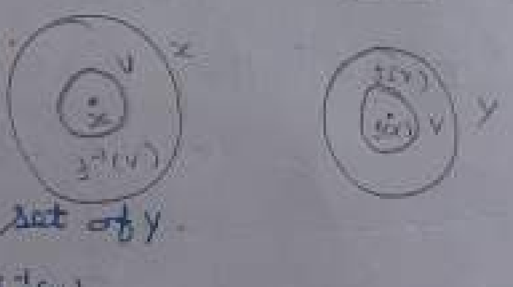
This is true for all open sets in Y.

$\Rightarrow f$ is Continuous.

To prove (i) $\Rightarrow$ (iv)

Let $x \in X$ and let V be a neighbourhood of $f(x)$ then the set $U = f^{-1}(V)$ is a $\cancel{f(A)}$ neigh. of x $\ni$:

$f(U) \subset V$.



To prove (iv) $\Rightarrow$ (i)

Let V be a open set of Y.

Let x be a point of $f^{-1}(V)$

$\Rightarrow x \in f^{-1}(V)$

Then $f(x) \in V$.

$\Rightarrow$ by hypothesis there is a neigh. U_x of x $\ni$: $f(U_x) \subset V$.

It follows that $f^{-1}(V)$ is written as Union of open sets U_x.

(i.e) $f^{-1}(V) = \bigcup_{x \in X} U_x$ is open in X.

(by a theorem).

(i.e) $f^{-1}(V)$ is open in X.

$\Rightarrow f$ is continuous.

Homomorphism

Let X and Y be topological spaces let $f: X \to Y$ be a bijective. If both the function f and the inverse fun. $f^{-1}: Y \to X$ are continuous then f is called homomorphism.

Remark:

The condition that f^{-1} be continuous says that for each open set U of X, the inverse image of U under the map $f^{-1}: Y \to X$ is open in Y. But the inverse image of U under the map f^{-1} is the same an the image of U under the map f. From the above diagram. $f: X \to Y$ is a bijective correspondent such that

$f(U)$ is open iff U is open.

Topological property of X

Any Property of X that is entirely expressed interms of the topology of X. [i.e Interms of the open sets of X] yields, Via the correspondence f, the corresponding property for the space Y.

Such a property of X is called the topological property of X.

Topological imbedding.

Suppose that $f : X \to Y$ is an injective continuous map, where X and are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y, then the fun. $f^{-1} : X \to Z$ obtaine by restricting the range of f is bijective. If f^{-1} happens to be a homomorphism of X with Z, then we say that $f : X \to Y$ is a topological imbedding of X in Y.

Example: (for homomorphism)

1. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x+1$ is a homomorphism.

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(y) = \tfrac{1}{3}(y-1)$$

To show that,

$$f(g(y)) = y$$

$$\begin{aligned} f(g(y)) &= f(\underbrace{\tfrac{1}{3}(y-1)}_{x}) \\ &= 3(\tfrac{1}{3}(y-1)) + 1 \\ &= y-1+1 \\ &= y \end{aligned}$$

To show that

$$g(f(x)) = x$$

$$\begin{aligned} g(f(x)) &= g(\underbrace{3x+1}_{y}) \\ &= \tfrac{1}{3}(3x+1-1) = x \end{aligned}$$

$\therefore f$ is bijective and $g = f^{-1}$.

The continuity of f and g is thus follows.

Theorem 2.2

15m ⑤ I-term

Rules for Constructing Continuous functions.

5m

Statement :

Let X, Y and Z be topological spaces

a) (constant function). If $f: X \to Y$ maps all of X into the single point y_0 of Y then f is continuous.

b) (Inclusion) If A is a subspace of X, the inclusion fun.

$j: A \to Y$ is Continuous.

c) (Composites) If $f: X \to Y$ and $g: Y \to Z$ are continuous, then the map $g \circ f: X \to Z$ is Continuous.

d) (Restricting the domain). If $f: X \to Y$ is Continuous, and if A is a subspace of X, then the restricted fun.

$f/A: A \to Y$ is Continuous.

e) (Restricting (or) expanding the range).

Let $f: X \to Y$ be Continuous. If Z is a subspace of Y containing the image set $f(X)$, then the fun. $g: X \to Z$ obtained by restricting the range of f is Continuous. If Z is a space having subspace, then the funs.

$h: X \to Z$ obtained by expanding the range of f is continuous.

f) (Local formulation of Continuity).

The map $f: X \to Y$ is Continuous if X can be written as the union of open sets U_α ∋: f/U_α is points for each α.

Proof :-

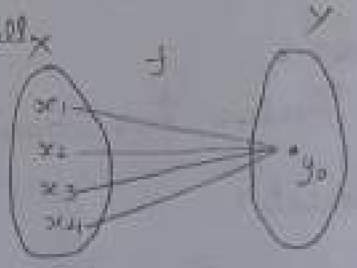
To prove (1) (a)

Let $f: X \to Y$ be such that

$f^{-1}(v) = X$, $f^{-1}(v)$ contains all points of X

$f(x) = y_0$ $\forall x \in X$

$\Rightarrow f(v) \in T$



Let V be open set in Y.

$\Rightarrow f^{-1}(v)$ is either X (or) ϕ depending on whether V contains y_0 (or) not

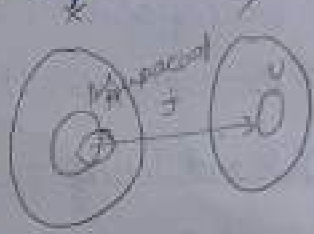
$\Rightarrow f^{-1}(v)$ is open in X.

$\therefore$ by defn. f is continuous.

Def of space topology
Y is subspace of them
$T_Y = \{U \cap Y \mid U$ is open in Y

To prove (2) (b)

Consider the inclusion map $j : A \to Y$.



A is a subspace of Y.

Let U be open in Y.

$\Rightarrow$ by defn of subspace topology

$$j^{-1}(U) = U \cap A \text{ is open in } A.$$

by the subspace topology T_A on A the fun. j is continuous.

To prove (3) c

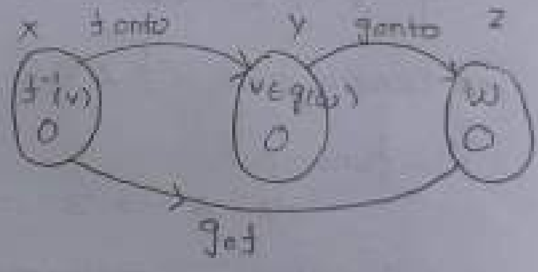
Let X, Y, Z be three spaces.

Let $f : X \to Y$ and $g : Y \to Z$ be two given funs.

without loss of generality

Assume that f & g be two onto funs.

(ie) $f(x) = y$, $g(y) = z$



Claim:

$g \circ f$ is continuous

w.k.T $\quad (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

choose any arbitrary open set W in Z.

$$(g \circ f)^{-1}(W) = f^{-1} \circ g^{-1}(W)$$
$$= f^{-1}(g^{-1}(W))$$
$$= f^{-1}(V)$$
where $V = g^{-1}(W)$

$(f \circ g)(x) = f(g(x))$
As f & g are continuous
$\Rightarrow g^{-1}(W$ is open in $Y)$
& $f^{-1}(V)$ is open in X

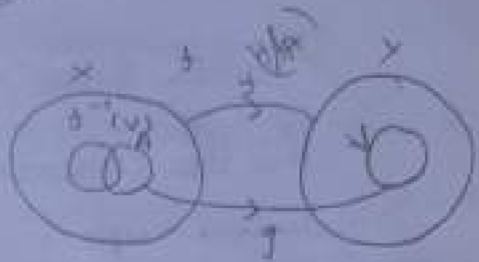
$\therefore (g \circ f)^{-1}(W)$ is open in X.

$\Rightarrow g \circ f$ is Continuous

To prove 4(d)

Continuous.

Let $f: X \to Y$ be ~~points~~ call $f/_A = g$.



As f is Continuous

$\Rightarrow$ ~~for~~ each open set V in Y

$f^{-1}(V)$ is open in A

To Prove:

$g: A \to Y$ is Continuous It is observed that f & g the identically as far as the subspace A is concerned.

For any open set V in Y,

$g^{-1}(V) = f^{-1}(V) \cap A$ open in A

$g^{-1}(V)$ is open in A.

$\therefore \tau_A = \{UA / U$ is open in $X\}$ by defn of subspace topology.

$\Rightarrow g$ is Continuous

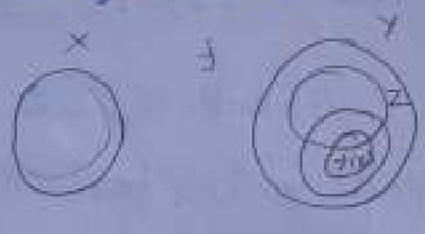
(i.e) $f/_A$ is Continuous

To prove (5) ~~e~~

Let $f: X \to Y$ be Continuous w.l

i) To show that: $g: X \to Z$, obtained from f is continuous.

Let $f(X) \subseteq Z \subseteq Y$.



Choose any open set B in Z, $\exists$..,

$B = U \cap Z$ when U is open in Y.

$g^{-1}(B) = g^{-1}(U \cap Z)$

$= g^{-1}(U) \cap g^{-1}(Z)$.

$= g^{-1}(U) \cap X$.

$= g^{-1}(U)$

$g^{-1}(V) \subseteq X$.

and $f(X) \subseteq Z$.

$g^{-1}(B) = f^{-1}(U)$

by Continuity of f.

$f^{-1}(U)$ is open in X.

$g^{-1}(B)$ is open in X.

(i.e) $g : X \to Z$ is Continuous.

(ii) To Show that

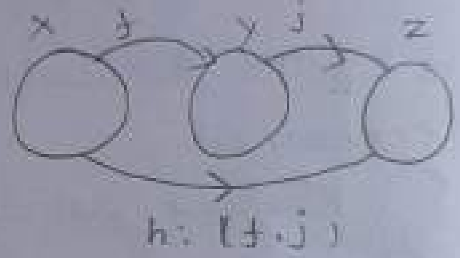
$h : X \to Z$ is Continuous

where Z has Y as a Subspace.

h is the composition of the maps $f : X \to Y$ and the inclusion maps $j : Y \to Z$.

Also, $f : X \to Y$ & $j : Y \to Z$ are continuous

by the fact the composition of maps is continuous.

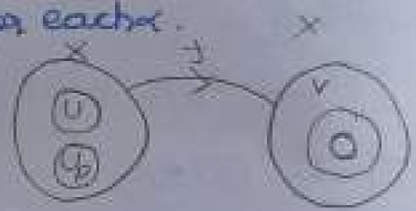


$\Rightarrow f \circ j$ is Continuous

$\Rightarrow h$ is continuous from X to Z.

To prove (b)

by data - let X be written as the union of open sets $U_\alpha \ni f/U_\alpha$ is Continuous for each α.



Claim:

$f : X \to Y$ is Continuous.

Let V be any open set in Y.

$f^{-1}(V) = V \cap Y$

Then $f^{-1}(V) \cap U_\alpha = (f/U_\alpha)^{-1}(V)$

As both the sets represents the sets of those points x lying in U_α for which $f(x) \in V$.

f/U_α is continuous.

$\Rightarrow (f/U_\alpha)^{-1}(V)$ is open in U_α & hence open in X.

But $f^{-1}(V) = \bigcup_{\alpha \in J} (f^{-1}(V) \cap U_\alpha)$ being arbitary union

$f^{-1}(V)$ is open in X for the open set V in Y.

$\therefore$ f is Continuous.

(17) 6m (5) 2-times

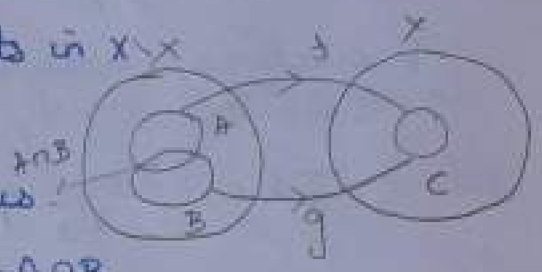
Theorem 2.3 (Pasting lemma) (6m) (x) (9) (M)

Statement: Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ $\forall\, x \in A \cap B$. Then f & g combine to give a continuous fun. $h : X \to Y$ defined by $h(x) = f(x)$ if $x \in A$
$= g(x)$ if $x \in B$.

Proof:-

Let $X = A \cup B$

where A & B are closed sets in X.



Let $f : A \to Y$.

$g : B \to Y$ be continuous.

Also let $f(x) = g(x)$ $\forall\, x \in A \cap B$.

Claim

$h : X \to Y$ is continuous where $h(x) = f(x)$ $\forall\, x \in A$.

Let C be any closed set in Y.

$$h^{-1}(C) = f^{-1}(C) \cap g^{-1}(C).$$

Here $f^{-1}(C)$ is closed in A and hence closed in X.

Also $g^{-1}(C)$ is closed in B and hence closed in X.

$\Rightarrow f^{-1}(C) \cap g^{-1}(C)$ is closed in X

$\Rightarrow h^{-1}(C)$ is closed in X.

$\therefore$ C is closed in $Y \Rightarrow h^{-1}(C)$ is closed in X

$\therefore h : X \to Y$ is continuous.

Theorem 2.4 (Maps into products) (8) I. Term (X) 6m (18)

Let $f : A \to X \times Y$ be given by $f(a) = (f_1(a)\ f_2(a))$, $a \in A$. Then f is continuous iff $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.

The maps f_1 & f_2 are called coordinate funs. of f.

Proof: Direct Part: $A \to X \times Y$

Assume $f : A \to Y$ be Continuous

claim: $f_1 : A \to X$, $f_2 : A \to Y$ are Continuous

Consider the projections.

$\pi_1 : X \times Y \to X$

$\pi_2 : X \times Y \to Y$ onto the first and second these

Projections are continuous for

$$\pi_1^{-1}(U) = U \times Y$$

$$\pi_2^{-1}(V) = X \times V$$

These sets are open if U & V are open in X & Y.

if $a \in A$, $f_1(a) = \pi_1(f(a))$

$f_2(a) = \pi_2(f(a))$

Suppose f is continuous being Composition of Continuous funs. f_1 & f_2 are Continuous.

Converse Part:

Let f_1, f_2 be continuous.

Claim: $f : X \to X \times Y$ is Continuous.

Let U be open in X & V be open in Y

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

~~As $(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$~~

As $f_1^{-1}(U)$ is open in A.

& $f_2^{-1}(V)$ is open in A.

$\Rightarrow f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in A

$\Rightarrow f^{-1}(U \times V)$ is open in A

$\Rightarrow f$ is Continuous.

The Product Topology.

Def:

Let $\mathcal{J}$ be an index set. Given a set X we define a $\mathcal{J}$-tuples of continuous of X to be a fun. $x : \mathcal{J} \to X$. $x : \mathcal{J} \to X$

If α is an continuous of $\mathcal{J}$ denote the value of x at α by x_α rather than $X(\alpha)$ call it as the α^{th} co-ordinate of X denote the fun. X itself by the symbol.

$$(x_\alpha)_{\alpha \in \mathcal{J}} \in \mathcal{J}$$

which is an close as tuples notatio for an arbitary index set $\mathcal{J}$ denote the set of all $\mathcal{J}$-tuples of continuous of X by $X^{\mathcal{J}}$.

Def:

Let $\{A_\alpha\}_{\alpha \in \mathcal{J}}$ be an indexed family of sets. Let $X = \bigcup_{\alpha \in \mathcal{J}} A_\alpha$. The Cartesian Product of thus indexed family denoted by $\prod_{\alpha \in \mathcal{J}} A_\alpha$.

is defined to be the set of all $\mathcal{J}$-tuples $(x_\alpha)_{\alpha \in \mathcal{J}}$ of continuous of X $\ni$: $x_\alpha \in A_\alpha$ for each $\alpha \in \mathcal{J}$.

That is it is the set of all functions

$(x_1) = (x_{11}, x_{12}, \dots, x_{1j})$

$(x_2) = (x_{21}, x_{22}, \dots x_{2j})$

$$x : \mathcal{J} \to \bigcup_{\alpha \in \mathcal{J}} A_\alpha$$

$\ni$: $x(\alpha) \in A_\alpha$ for each $\alpha \in \mathcal{J}$.

Occasionally denote the product simply by $\prod A_\alpha$, and its general continuous by (x_α) if the index set is understood.

Def:

Let $\{X_\alpha\}_{\alpha \in \mathcal{J}}$ be an indexed family of topological spaces take as a basis for a topology on the product space

$$\prod_{\alpha \in \mathcal{J}} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} U_\alpha$$

Where U_α is open in X_α for each $\alpha \in J$ the topology generated by this basis is called the box - topology.

This collection satisfies the 1st condition for a basis because ΠX_α is itself a basis continious and it satisfies the 2nd condition because the intersection of any two basis continious is another basis continious.

$$\boxed{\left(\prod_{\alpha \in J} U_\alpha\right) \cap \left(\prod_{\alpha \in J} V_\alpha\right) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)}$$

Now, generalize the sub basis formulation of the definition. let $\boxed{\pi_\beta (x_\alpha)_{\alpha \in J} = x_\beta.}$

it is called the projection mapping associated with the index β.

Defn:

Let S_β denote the collection.

$$S_\beta = \{ \pi_\beta^{-1} (U_\beta) \cdot U_\beta \text{ open in } X_\beta \}$$

and let S denote the union of iterm collections.

$$S = \bigcup_{\beta \in J} S_\beta$$

The topology generated by the subbasis S is called the Product topology.

In this topology $\prod_{\alpha \in J} X_\alpha$ is called a Product space.

Theorem 2.5 Comparision of the box and Product topologies.

The box topology on ΠX_α has basis all sets of the form ΠU_α . where U_α is open in X_α for each α.

The product topology on $\prod X_\alpha$ has an basis all set of the form $\prod U_\alpha$. where U_α is open in X_α for each α & U_α equals X_α except for finitely many values of α.

Proof:-

To Compare the box Product topologies consider the basis $\mathcal{B}$ that S' generates the collections of all finite intersections of element of S. As

$$\pi_\beta^{-1}(U_\beta)\, \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta).$$

The intersection of two. element of S_β or of finitely many such ____ again an element of S_β.

The typical element of the basis $\mathcal{B}$ can thus be described as follows.

Let $\beta_1, \beta_2 \dots \beta_n$ be a finite set of distinct indices from the index set J and let U_{β_i} be an open set in X_{β_i} for $i = 1, 2 \dots n$

Then

$$B = \pi_{\beta_1}^{-1}(U\beta_1) \cap \pi_{\beta_2}^{-1}(U\beta_2) \cap \dots \cap \pi_{\beta_n}^{-1}(U\beta_n) \text{ is}$$

the typical element of $\mathcal{B}$.

Now a point $x = (x_\alpha)$ is in B iff its β_1^{th} co-ordinate is in U_{β_1} its β_2^{th} co-ordinate is in U_{β_2} and so on.

There is no restriction whatever on the α^{th} co-ordinate of x if α is not or of the indices $\beta_1, \beta_2 \dots \beta_n$. As a result.

we can write B as the Product

$$B = \prod_{\alpha \in J} U_\alpha.$$

where U_α element the entire space X_α is

$$\alpha \neq \beta_1, \beta_2 \dots \beta_n.$$

Theorem 2.6.

Suppose the topology on each space A_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form $\prod_{\alpha \in J} B_\alpha$, where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology $\prod_{\alpha \in J} X_\alpha$.

Theorem : 2.7

Let A_α be a subspace of X_α for each $\alpha \in J$. Then $\prod_{\alpha \in J} A_\alpha$ is a subspace of $\prod X_\alpha$ if both Products are given the box topology (or) if both Products are given the product topology.

Theorem 2.8

If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.

Theorem 2.9

Let (X_α) be an indexed family of space. Let $A_\alpha \subset X_\alpha$ for each α if $\prod X_\alpha$ given either the Product (or) the box topology. Then

$$\prod \overline{A}_\alpha = \overline{\prod A_\alpha}$$

Proof:- Let $x = (x_\alpha)$ be a point of $\prod \overline{A}_\alpha$.

To Show that $x \in \overline{\prod A_\alpha}$

Let $U = \prod U_\alpha$ be a basis element for either the box or product topology that contains x.

$\because$ $x_\alpha \in \overline{A}_\alpha$, choose a point $y_\alpha \in U_\alpha \cap A_\alpha$ for each α.

Then $y = (y_\alpha)$ belongs to both U & $\prod A_\alpha$

$\because$ U is arbitary it follows that x belongs to the closure of πA_α.

Conversely.

Suppose $x = (x_\alpha)$ lies in the closure of πA_α in either topology.

To Show that for any given index β. $x_\beta \in \bar{A}_\beta$.

Let V_β be an arbitary open set of X_β containing x_β.

$\because \pi_\beta^{-1}(V_\beta)$ is open in πX_α in either topology it contains a point $y = y_\alpha$ of πA_α.

Then y_β belongs to $U_\beta \cap A_\beta$ it follows that $x_\beta \in \bar{A}_\beta$.

Theorem 2.10

Let $f : A \to \pi_{\alpha \in I} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in I}$ where $f_\alpha : A \to X_\alpha$ for each α. Let πX_α have the Product topology the fun. of f is continuous iff each fun. f_α is continuous.

Proof:- Let π_β be the projection of $\pi_{\alpha \in I} X_\alpha$ on to its β^{th} factors X_β.

The Projection π_β is continuous for if V_β is open in X_β.

Then $\pi_\beta^{-1}(V_\beta)$ is a subbasic open set for the product topology on πX_α. (a)

Now,

Suppose that $f : A \to \pi_{\alpha \in I} X_\alpha$ given by

$f(a) = (f_\alpha(a))_{\alpha \in I}$, for each '$\alpha$'. where $f_\alpha : A \to X_\alpha$ for each 'α' is continuous the functions.

$$f_\beta = \pi_\beta \circ f.$$

If are assume f in continuous, $\pi_\beta \circ f$ is continuous.

(i.e) f_β is continuous for $\beta \in I$.

To Prove the converse, let each co-ordinate fun. f_α be continuous.

Claim: f is continuous it is sufficient to S.T

f^{-1} of each subbasic open set open in A.

In $\prod_{\alpha \in J} X_\alpha$ choose a subbasic open set of the form $\pi_\beta^{-1}(V_\beta)$ where β is some index and U_β is open in X_β.

Now, $f^{-1}(\pi_\beta^{-1}(V_\beta)) = f_\beta^{-1}(U_\beta)$

because $f_\beta = \pi_\beta$ of the set on the right is open in X_β $\because$ (f_β is continuous function) $\pi_\beta^{-1}(U_\beta)$ is open in X.

$\therefore$ The set on the left is open (i.e) f^{-1} of an open set in X_β is open on X.

(i.e) f is continuous the proof of the theorem

The metric Topology:

Defn: A metric on a set X is a function $d: X \times X \to \mathbb{R}$ having the following properties.

1) $d(x,y) \geq 0$ for all $x, y \in X$; equality holds if & only if $x=y$.

2) $d(x,y) = d(y,x)$ $\forall$ $x,y \in X$.

3) (Triangle inequality)

$d(x,y) + d(y,z) \geq d(x,z)$ $\forall$ $x,y,z \in X$.

Given a metric d on X, the number $d(x,y)$ is often called the distance between x & y in the metric d given $\epsilon > 0$

Consider the set.

$B_d(x,\epsilon) = \{y / d(x,y) < \epsilon\}$ of all points y whose distance from x is less than ϵ, it is called the ϵ-ball centered at x.

Defn:

It d is a metric on the set X then the collection of all ϵ-baths $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$ is basis for a topology on X called the metric topology induced by d.

The 1^{st} condition for a basis is trivial.

$\because$ $x \in B(x, \epsilon)$ for any $\epsilon > 0$ before checking the second condition for a basis we show that if y is a point of the basis element $B(x, \epsilon)$ then there is a basis element $B(y, \delta)$ centre at y that is contained in $B(x, \epsilon)$

Define δ to be the +ve numbers $\epsilon - d(x, y)$

Then $B(y, \delta) \subset B(x, \epsilon)$ for if $z \in B(y, \delta)$ then $d(y, z) < \epsilon - d(x, y)$ from which we conclude that

$d(x, z) \leq d(x, y) + d(y, z) < \epsilon$.

A set U is open in the metric topology induced by d iff for each $y \in U$, there is a $\delta > 0$, $\ni$: $B_d(y, \delta) \subset U$.

Now to check the second condition for a basis, let B_1 & B_2 be two basis element and let $y \in B_1 \cap B_2$.

we have j not shown that we can choose +ve numbers δ_1 & we conclude that $B(y, \delta) \subset B_1 \cap B_2$.

clearly, this condition implies that U is open.

Conversly, if U is open, it contains a basis element $B = B_d(x, \epsilon)$ containing y & B in turn contains a basis element $B_d(y, \delta)$ centered at y.

Metric topology:

Example ①:

Given a set X, define.

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The topology it induces is the discrete topology. The basis element is $B(x,1)$

$$B(x,1) = \{ y / d(x,y) < 1 \}.$$

Example : 2

The Standard metric on the real numbers $\mathbb{R}$ is defined by the equation,

$$d(x,y) = |x-y|.$$

Metrizable space :

If X is a topological space X is said to be metrizable if $\exists$ a metric d on the set X that induces the topology of X. A metric space is a metrizable space X together with a specific metric d that gives the topology of X.

Defn:

Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number $M \ni$:

$d(a_1, a_2) \leq M$ for every pair a_1, a_2 of points of A.

If A is bounded and non-empty the diameter of A is defined to be the number.

$$\text{dia } A = \text{Sup} \{ d(a_1, a_2) / a_1, a_2 \in A \}.$$

Theorem 2.11 (5m) (R) S 2-times

Let X be a metric space will metric d define $\bar{d} : X \times X \to \mathbb{R}$ by the eqn $\bar{d}(x,y) = \min\{d(x,y), 1\}$. Then $\bar{d}$ is a metric that induces the same topology as d. The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof: Let (X,d) be a metric space.

Define $\bar{d}(x,y) = \min\{d(x,y), 1\}$ $\forall\ x,y \in X$.

clearly $\bar{d}$ is a metric on X.

For (1) $\bar{d}(x,y) \geq 0$ for $\forall\ x,y \in X$

Because, $d(x,y) \geq 0$ for $\forall\ x,y \in X$ and

$$\bar{d}(x,y) = 0 \Rightarrow x = y$$

$$d(x,y) = 0 \text{ if } x = y$$

2) $\bar{d}(x,y) = \min\{d(x,y), 1\}$

$= \min\{d(y,x), 1\}$

$= \bar{d}(y,x)$ $\forall\ x,y \in X$.

To check me triangle inequality $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

Now, either $d(x,y) \geq 1$ (or)

$d(y,z) \geq 1$

Then the R.H.S of the above inequality is atleast 1.

$\because$ L.H.S is atmost '1'.

The inequality holds. It remains to consider the case in which $d(x,y) < 1$, & $d(y,z) < 1$.

In this case $d(x,z) \leq d(x,y) + d(y,z)$

$= \bar{d}(x,y) + \bar{d}(y,z)$

From defn. $\bar{d}(x,z) \leq d(x,z)$

$\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

Proving the triangle inequality The facts that the metrices d and $\bar{d}$ induce the some topology is a consequence the following inclusion.

$$B_d(x : \epsilon) \subseteq B_{\bar{d}}(x ; \epsilon)$$

and $B_{\bar{d}}(x : \delta) \subseteq B_d(x : \epsilon)$

where $\delta = \min\{\epsilon, 1\}$

Hence proved.

Defn:

Given $x = (x_1, x_2 \ldots x_n)$ in $\mathbb{R}$ define the norm of x by the eqn.

$$\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$$

define the euclidean metric d on $\mathbb{R}$ by the eqn.

$$d(x,y) = \|x-y\| = [(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2]^{1/2}$$

define the square metric ρ by the equation.

$$\rho(x,y) = \text{Max}\{|x_1-y_1|, |x_2-y_2|, \cdots |x_n-y_n|\}$$

Lemma: 2.1

Let d and d' be two metrices on the set X. Let τ and τ' be the topologies they induce, respectively the τ' is finer than τ iff for each x in X and each $\varepsilon > 0$ $\exists$ a $\delta > 0$ $\ni$:

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon).$$

Proof:

Let τ' be finer than τ.

choose any basic open set $B_d(x; t)$ in τ.

then by a (lemma ③ unit I) $\exists$ a basic open set B' for topology τ' $\ni$: $x \in B' \subseteq B_d(x; t)$ with in B'.

We can find a ball $B_{d'}(x; \delta)$ centered at x and

$$\delta = \min\{\epsilon, 1\}$$

(i.e) $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

To prove the converse suppose $\delta-\epsilon$ condition hold (Given a basic open set B for τ containing x).

We can find with in B an open ball $B_d(x; \epsilon)$ centred at x.

$\therefore$ By the condition $\exists$ a $\delta > 0$.

$\ni$: $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

Apply a lemma 1.3 we get τ' is finer than τ.

Theorem 2.12 15m 5 18 2/km

The topologies on R^n induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof:- Consider $\bar{x} = (x_1, x_2, \dots x_n)$

$\bar{y} = (y_1, y_2, \dots y_n)$

we have from defn.

$\rho(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{y}) \leq \sqrt{n}\, \rho(\bar{x}, \bar{y})$

The 1st inequality

$\Rightarrow$ That $B_d(\bar{x}, \epsilon) \subseteq B_\rho(\bar{x}, \epsilon)$ $\forall \bar{x} \in R^n$ & $\epsilon \geq 0$

Because $d(\bar{x}, \bar{y}) < \epsilon$

Then $\rho(\bar{x}, \bar{y}) < \epsilon$ also

Ill^{ly} The 2nd inequality.

$\Rightarrow$ That $B_\rho(\bar{x}, \epsilon/\sqrt{n}) \subseteq B_d(\bar{x}, \epsilon)$ $\forall \bar{x} \in R^n$ & $\epsilon \geq 0$.

$\therefore$ by Proceeding lemma.

That two metric topologies are identical

To S.T the Product topologes is the same as that given by the metric 'ρ'

First consider the basic open set.

$B = (a_1, b_1) \times, \dots (a_n, b_n) \times$

For the product topology.

Let $\bar{x} = (x_1, x_2 \dots x_n)$ be an elt in B.

Then for each i, it is possible find an $\varepsilon_i > 0$,

$\Rightarrow: (x_i - \varepsilon_i,\ x_i + \varepsilon_i) \subseteq (a_i, b_i)$

Now, choose

$\epsilon = \min\{\epsilon_1, \epsilon_2, \dots \epsilon_n\}$

Then,

$$B_\rho(\bar{x}, \epsilon) \in B.$$

$\therefore$ The ρ-topology to finer than product topology.

To prove the other way.

choose $B_\rho(\bar{x}, \epsilon)$ as a basic open set for the ρ-topology.

Given $\bar{y} \in B_\rho(\bar{x}, \epsilon)$ we need to determine the basic open set B for the product topology $\ni$: $\bar{y} \in B \in B_\rho(\bar{x}, \epsilon)$

But this is trivial for $(x_n - \varepsilon, x_n + \varepsilon)$

$B_\rho(\bar{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon) x_{n+\epsilon}$

is itself a basic open set for the product topology.

This is the product topology is finer than the ρ-topology

Then two together.

$\Rightarrow$ that the product coincides with ρ-topology

Hence proved.

Defn:

Given an index set τ, and given points $x = (x_\alpha)_{\alpha \in \tau}$ and $y = (y_\alpha)_{\alpha \in \tau}$ of $\mathbb{R}^\tau$.

define a metric $\bar{\rho}$ on $\mathbb{R}^\tau$ by the eqn.

$\bar{\rho}(x, y) = \operatorname{Sup}\{\bar{d}(x_\alpha, y_\alpha) / \alpha \in \tau\}$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. It is called the unifor metric on $\mathbb{R}^\tau$ and the topology $(\bar{\rho})$.

it induces is called the uniform topology.

Theorem 2.13

The uniform topology on R^τ is finer than the Product topology and coarser than the box topology, then three topologies are all different if τ is infinite.

Proof:

Suppose that we are given a point $x = (x_\alpha)_{\alpha \in \tau}$ and a product topology basis element $\prod U_\alpha$ about x.

Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be the indices for which $U_\alpha \neq R$.

Then for each i, choose $\epsilon_i > 0$ So that the ϵ_i-ball Centered at x_{α_i} in the $\bar{d}$ metric in Contained in U_{α_i}, this we can choose so because U_{α_i} is open in R.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2 \ldots \epsilon_n\}$ then the ϵ-ball centred at x in the $\bar{P}$ metric is Contained in $\prod U_\alpha$.

For if z is a point of $R^\tau \ni$: $\bar{P}(x, z) < \epsilon$, then $\bar{d}(x_\alpha, z_\alpha) < \epsilon \ \forall \alpha$, so that $z \in \prod U_\alpha$.

It follows that the uniform topology is finer than the product topology.

on the other hand, let B be the ϵ-ball Centered at x in the $\bar{P}$ metric.

Then the box neighbourhood

$U = \prod(x_\alpha - \frac{1}{2}\epsilon, x_\alpha + \frac{1}{2}\epsilon)$ of x is Contained in B.

For if $y \in U$ then $\bar{d}(x_\alpha, y_\alpha) < \frac{1}{2}\epsilon \ \forall \alpha$,

So that $\bar{P}(x, y) \leq \frac{1}{2}\epsilon$

Hence proved.

Theorem 2.14

Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x & y are two points of $\mathbb{R}^{\omega}$ define,

$$D(x,y) = \sup\left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^{\omega}$.

Proof:-

The Properties of a metric are satisfied trivially except for the triangle inequality, which is Product Proved by nothing that for all i

$$\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i} \leq D(x,y) + D(y,z)$$

So that,

$$\sup\left\{ \frac{\bar{d}(x_i, z_i)}{i} \right\} \leq D(x,y) + D(y,z)$$

The fact that D gives the product topology requires a little more work.

First, let U be open in the metric topology and let $x \in U$.

We find an open set V in the product topology $\ni$: $x \in V \subset U$.

Choose an ϵ-ball $B_D(x, \epsilon)$ lying in U. Then choose N large enough that $\frac{1}{N} < \epsilon$.

Finally, let V be the basis for the Product topology.

$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$

We ~~asser~~ assorts that $V \subset B_D(x, \epsilon)$.

Given any y in $\mathbb{R}^{\omega}$.

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N} \text{ for } i \geq N.$$

$$D(x,y) \leq \max\left\{ \frac{\bar{d}(x_1, y_1)}{1} \cdots \frac{\bar{d}(x_N, y_N)}{N} \frac{1}{N} \right\}$$

If y is in V, this expression is less than ϵ, So that $V \subset B_D(x, \epsilon)$ as desired.

Conversly,

consider a basis element $U = \prod_{i \in \mathbb{Z}_+} U_i$ for the Product topology, where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \ldots \alpha_n$ and $U_i = \mathbb{R}$ for all other indices i. Given $x \in U$, we find an open set V of the metric topology $\ni: x \in V \subset U$.

Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R}$ centred about x_i and lying in U_i for $i = \alpha_1, \alpha_2 \ldots \alpha_n$, choosen each $\epsilon_i \leq 1$.

Then define $\epsilon = \min\{\epsilon_i / i, i = \alpha_1, \alpha_2 \cdots \alpha_n\}$

we ~~assert~~ that $x \in B_D(x, \epsilon) \subset U$.

Let y be a point of $B_D(x, \epsilon)$ then $\forall i$

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x,y) < \epsilon$$

Now if $i = \alpha_1, \alpha_2 \ldots \alpha_n$ then $\epsilon \leq \epsilon_i / i$

So that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$, it follows

then $|x_i, y_i| < \epsilon_i$

$y \in \prod U_i$ as desired.

Metric topology (continued)

Theorem 2.15

Let $f: X \to Y$. Let X and Y be metrizable with metric d_X and d_Y respective then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, $\exists\ \delta > 0 \ni$:

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon.$$

Proof:

Direct Part:

Suppose f is continuous.

Given x & ϵ, Consider the set $f^{-1}(B(f(x), \epsilon))$ which is open in X and contains x.

It contains some δ-ball $B(x, \delta)$ centered at x. If y is in this δ-ball, then $f(y)$ is in the ϵ-ball centered at $f(x)$ as desired conversely.

Suppose that the ϵ-δ condition is satisfied.

Let V be open in Y, we show that $f^{-1}(V)$ is open in X.

Let x be a point of the set $f^{-1}(V)$. Since $f(x) \in V$, then is an ϵ-ball $B(f(x), \epsilon)$ centered at $f(x)$ and contained in V - by the ϵ-δ condition, there is a δ-ball $B(x, \delta)$ centered at $x \ni$:

$$f(B(x, \delta)) \subset B(f(x), \epsilon)$$

Then $B(x, \delta)$ is a nbd of x contained in $f^{-1}(V)$. So that $f^{-1}(V)$ is open as desired.

Hence Proved.

Defn:

A sequence $(x_1, x_2, \dots)$ of the point in X is said to converge to the point $x \in X$ if for every nbd U of x, $\exists$ a +ve integer N. $\ni$: x_i lies in U $\forall\ i \geq N$.

Lemma 2.2 Sequence lemma.

Let X be a topological space. Let $A \subset X$. If there is a sequence of points in A converging to x. then $x \in \bar{A}$, The converse holds if X is metrizable.

Proof:-

Direct Part:

Suppose $x_n \to x$, where $x_n \in A$ From defn (of convergence) U of x contains a point of A.

(i.e) $x \in \bar{A}$

Converse part:

Let $x \in \bar{A}$, let 'd' be the metric for the topology on X. For each 'n' choose a nbh $B_d(x, 1/n)$ of the point x and choose x_n to be a point in $B_d(x, 1/n)^n$

To Prove: $x_n \to x$ $\forall n$.

Now any open set U containing x contians on ϵ-ball $B_d(x, \epsilon)$ around x Choose N. So that $1/N < \epsilon$, then U contains x, for $i \geq N$.

(i.e) $x_n \to x$.

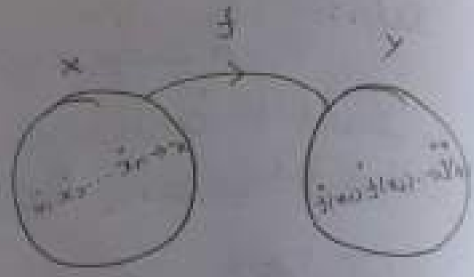
The sequence $x_1, x_2 \dots$ of points of A converges to x.

Theorem 2.16 Let $f: X \to Y$ be a function. Let X be metrizable. The fun f is continuous iff for every convergent sequence $x_n \to x$ in X. The sequence $f(x_n) \to f(x)$.

15 mark

Proof:

Direct part



Let $f: X \to Y$ be continuous

Suppose in X. $x_n \to x$.

Claim: $f(x_n) \to f(x)$

Let V be a nbd of $f(x)$

f is continuous $\Rightarrow f^{-1}(V)$ is open in X and

hence it is a nbh of x.

$\therefore$ by data $\exists$ a stage N, $\ni$:

$x_n \in f^{-1}(V)$ for $n \geq N$.

$\Rightarrow f(x_n) \in V$ for $n \geq N$

$\Rightarrow f(x_n) \Rightarrow f(x)$

Converse part:

Suppose $x_n \to x \Rightarrow f(x_n) \to f(x)$

Claim: f is continuous

Let $A \subset X$, choose any point $x \in \bar{A}$

$\Rightarrow f(x) \in f(\bar{A})$ — (1)

by sequence lemma $\{x \in \bar{A}$ iff $x_n \to x$

$\exists$ a sequence of points x_n in A Converges to x.

by hypothesis $f(x_n) \to f(x)$.

$\therefore f(x_n) \in f(A)$

Again by sequence lemma

$f(x) \in \overline{f(A)}$ — (2)

$\therefore A \subset \bar{A}\}$

$f(A) \subset \overline{f(A)}$

From (1) & (2)

$f(\bar{A}) \subset \overline{f(A)}$

by theorem (2.1).

$\therefore f$ is continuous.

Defn:

Let $f_n : X \to Y$ be a sequence of funs. from the set X to the metric space Y. Let d be the metric for Y. The sequence (f_n) converges uniformly to the function $f : X \to Y$ if given $\epsilon > 0$ there exists an integer $N \ni$:

$$d(f_n(x), f(x)) < \epsilon \quad \forall\ n \geq N \text{ and all } x \text{ in } X.$$

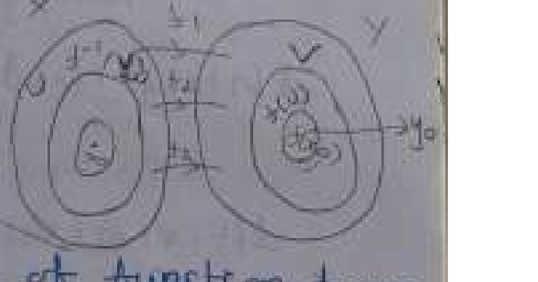
$f(x_n) \to f(x)$
V be a nbh $f(x)$ of
continuous $f^{-1}(V)$ is open
$x_n \in f^{-1}(V)$
$f(x_n) \in V$
$f(x_n) \to f(x)$

$f(x_n) \Rightarrow f(x)$
claim f in continuous
$A \subset X \quad x \in \bar{A} \quad f(x) \in f(\bar{A})$
$f(x_n) \to f(x)$
$f(x_n) \Rightarrow f(A)$
$f(x) \in \overline{f(A)}$
$f(\bar{A}) \subset \overline{f(A)}$

n x S 15m

Theorem 2.17 : Uniform limit theorem

Statement : Let $f_n : X \to Y$ be a sequence of continuous funs. from the space X to the metric space (Y, d). If function converges uniformly to f. then f is continuous.



Proof:

To prove:

$f_n \to f$ and f is continuous.

Let $f_n : X \to Y$ be a sequence of continuous function from space X into metric space.

Let f_n converges uniformly to f.

Claim:

f is continuous.

To this end choose any open set V in Y.

Let $x \in f^{-1}(V)$

It remains to show that $\exists$ a nbh U of x_0 so that $f(U) \subset V$.

Call $y_0 = f(x_0)$. First choose ϵ so that the ϵ-ball $B(y_0, \epsilon) \subset V$ as V is open.

Apply uniform converges choose a natural number $N \ni$:

$d(f_n(x), f(x)) < \epsilon/4 \quad \forall n \geq N$ and $\forall x \in X$.

Then as f_N is continuous, we can choose a nbh of x_0 $\ni$: The function f_N carries U into the $\epsilon/2$-ball in Y centered at $f_N(x)$.

we now S.T f carries U into $B(y_0, \epsilon)$ and hence into V as required.

To a achieve this if $x \in U$ then,

$d(f(x), f_N(x)) < \epsilon/4$, by the choice of N.

$d(f_N(x), f_N(x_0)) < \epsilon/2$, by the choice of U.

$d(f_N(x_0), f(x_0)) < \epsilon/4$, by the choice of N.

by applying,

Triangle inequality

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0))$$

$$d(f(x), f(x_0)) < \epsilon/4 + \epsilon/2 + \epsilon/4$$

$$= \epsilon \text{ for } \forall x \in U.$$

$\therefore f$ is continuous at x_0.

x_0 is arbitrary

$\therefore f$ is continuous on X.

Hence Proved.

Unit - III

Connectedness.

Defn: Connectedness:

Let X be a topological space. A separation of X is a pair U, V of disjoint non empty open subsets of X whose union equals X.

The space X is said to be connected if there does not excist a separation of X.

6m **Theorem 3.1** 16 S 6m

A space X is connected iff the only subsets of X that are both open & closed in X are the empty set and X itself.

Proof:

Let A be a non-empty proper subset of X which is both open & closed in X.

Claim:

X is not connected.

U = A, which is open in X and V = X - A, also open in X. A is closed

∴ U is a non-empty Proper subset of X, V is also non-empty.

Further U & V are disjoint and this union equals X.

Thus U & V from a separation X.

(i.e) X is not connected.

Conversely, X is not connected.

(i.e) U & V form a separation of X.

So, U is open and non-empty. Also, U = X - V is closed in X.

Also U is a proper subset of X.

(∵ V is non-empty)

Hence we are able to find U as a non-empty proper subset of X which is both open and closed in X.

Hence proved

Lemma 3.1

If Y is a subspace of X, a separation of Y is a pair of disjoint non-empty sets A & B whose union is Y, neither of which contains a limit point of the other the space Y is connected if $\exists$ no separation of Y.

Proof.

Suppose A and B from a separation of Y.

claim:

$$A \cap \overline{B} = \phi$$
$$\overline{A} \cap B = \phi$$

using the Previous theorem we can find 'A' to be a non-empty proper subset of Y which is both open and closed in Y.

$\because$ $\overline{A}$ denote the closure of A in X & Y being a subspace X Then $\overline{A} \cap Y$ is a closure of A in Y.

$\because$ A is closed in Y, by a theorem, we have.

$$A = \overline{A} \cap Y.$$

Now $A \cap B = (\overline{A} \cap Y) \cap B$.

$$= \overline{A} \cap (Y \cap B).$$

(i.e) $\phi = \overline{A} \cap B$

$\|^{ly}$ we can claim that

$$A \cap \overline{B} = \phi$$

Conversely:

Suppose A & B are non-empty, and B are disjoint $A \cup B = Y$ and $A \cap \overline{B} = \phi$, $\overline{A} \cap B = \phi$

claim

A and B form a separation of Y. It is enough to show that A & B are open in Y.

consider,

$$A \cap \overline{B} = \phi$$

$$B \cup (A \cap \overline{B}) = B \cup \phi$$

$$(B \cup A) \cap (B \cup \overline{B}) = B \cup \phi = B$$

$$Y \cap \overline{B} = B \quad (\because B \subset \overline{B})$$

by a, theorem,

B is closed in Y.

then $A = Y - B$ which is open in Y.

consider,

$$\overline{A} \cap B = \phi$$

$$A \cup (\overline{A} \cap B) = A \cup \phi$$

$$(A \cup \overline{A}) \cap (A \cup B) = A$$

$$\overline{A} \cap Y = A$$

by a thm,

A is closed in Y.

Hence $B = Y - A$, which is open in Y.

Lemma 3.2

If the sets C & D form a separation of X and if Y is a connected subspace of X. Then Y lies entirely with C or D

Proof:-

$\because$ C and D form a separation of X, C & D are open in X. since Y being a subspace of X.

we have,

$C \cap Y$ and $D \cap Y$ are open in Y.

[Refer from the defn. of subspace].

Also they are disjoint and their union equals Y.

(i.e) $(C \cap Y) \cup (D \cap Y) = Y$ — (1)

If they were both non-empty then they form a separation of Y
which is $\Rightarrow\Leftarrow$

The fact that Y is connected.

$\therefore$ one of the two sets must be empty.

[either $C \cap Y = \phi$ (or) $D \cap Y = \phi$]

Suppose $D \cap Y = \phi$

Then $C \cap Y = Y$ (from ①)

Hence $Y \subset C$

Hence the theorem.

Theorem 3.2

(*) The union of a collection of connected subspace of X that have a point in common in connected.

Proof:-

Let $\{A_\alpha\}$ be a collection of connected subsets of a space X, let $P \in \bigcap_{\alpha \in I} A_\alpha$.

Claim

$Y = \bigcup_{\alpha \in I} A_\alpha$ is connected.

Suppose that $Y = C \cup D$ form a separation of Y then the point P is in one of the two sets C or D

suppose point $P \in C$

$\because A_\alpha$ is a connected subset of Y and $Y = C \cup D$ form a separation of Y. using Previous theorem, A_α must lie entirely with in C (or) D

But A_α Cann't lie in D. because it Contains a point

($\because$ C & D are disjoint)

Thus $A_\alpha \subset \mathcal{C}$ This is true for every α.

$\therefore \bigcup_{\alpha \in I} A_\alpha \subset \mathcal{C}$

(i.e) $Y \subset \mathcal{C}$ which $\Rightarrow\!\Leftarrow$ the fact that $\mathcal{D}$ is non-empty.

Hence Y is connected.

Theorem 3.3

Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$. Then B is also connected.

Proof:-

Let A be a connected ~~subset~~ space of X and Let $A \subset B \subset \overline{A}$ (given).

Claim:- B is connected suppose $B = C \cup D$ for a separation of B.

(By a lemma 3.2)

A must then entirely within C or D.

[$\because$ C & D forms a separation of B and A is a connected subset of B

Then A must lie entirely with in C or D]

Suppose $A \subset C$

Then $\overline{A} \subset \overline{C}$

$\therefore B \subset \overline{A} \subset \overline{C}$ ($\because B \subset \overline{A}$)

by a lemma 3.1

we say that $\overline{C}$ and D are disjoint.

[A separation of B exists iff C and D are non-empty C and D are open in B

$C \cup D = B$ and

$\overline{C} \cap D = \phi$

$C \cap \overline{D} = \phi$].

∴ B ⊂ C, B and D are also disjoint.

(i.e) $B \cap D = S$ is empty. (∵ $B \cap D$ is non-empty)

But, From the defn of separation of B, we say $B \cap D$ is non empty.

Hence the contradiction arises because of our assumption so, B is connected.

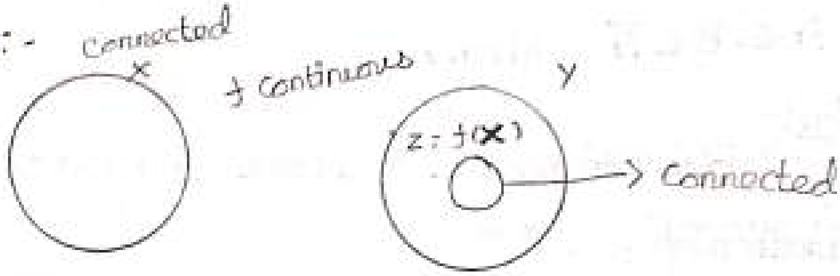
2 time (7) 6m S (8)

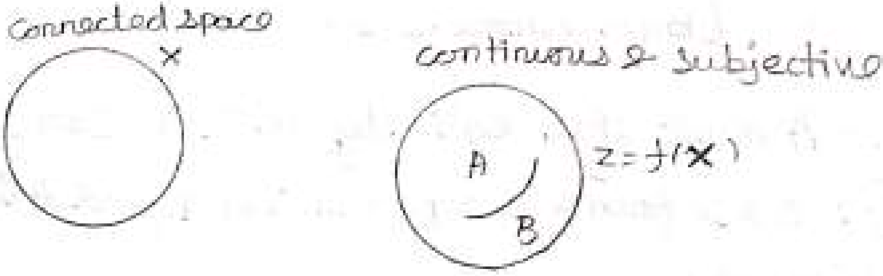
Theorem 3.4.

(*) The image of a connected space under a continuous map is connected.

6m (8)

Proof:-





Let $f : X \to Y$ be a continuous map. Let X be a connected space.

<u>claim</u> :

f(X) is connected

∵ the map obtained from "f" by restricting the range of "f" to the space z is also continuous.

[Refer chapter .2](sec-7)

It is enough to consider the case of a continuous subjective map $g : X \to Z$.

... fact that X is connected ... of X

Hence $Z = f(X)$ is connected.

Hence Proved.

Connected subspace of the Real line:

Defn:

A simply ordered set L having more than one element is called a linear continuous.

If the following hold:

i) L has the l.u.b property.

ii) If $x < y$, then there exists $z \ni: x < z < y$.

Theorem 3.5 15m 5 18 16 2 time 15m M 8

15mark. 15m

If L is a linear continuous in the order topology. Then L is connected and so are intervals & rays in L.

Proof:

Let Y be a subset of L that equals either L (or) an intervals (or) a ray in L.

[The set Y is convex in the sense that if a & b are any two points in Y and if $a < b$ then the entire interval $[a, b]$ of points of L is contained in this set Y].

Claim 1:

Y is connected.

Let A and B be two disjoint non-empty sets that are open in Y. We shall show that $Y \neq A \cup B$.

There by saying that there exists no separation of Y.

Choose a point 'a' in A & b in B, assume that notation so choosen that ~~notation so~~ $a < b$. because Y is convex we have $[a,b] \subset Y$.

Claim 2:

We shall find a point an $[a,b]$ that belongs to neither in A nor in B.

$$[a,b] \subset Y$$

$$[a,b] \neq A \cup B$$

$$Y \neq A \cup B$$

Consider a set $A_0 = A \cap [a,b]$

$B_0 = B \cap [a,b]$

These sets are open in $[a,b]$ in the subspace topology.

$$A_0 = A \cap [a,b]$$

$$B_0 = B \cap [a,b]$$

Further $a \in A_0$ and $b \in B_0$

$\therefore$ A_0 and B_0 are non-empty.

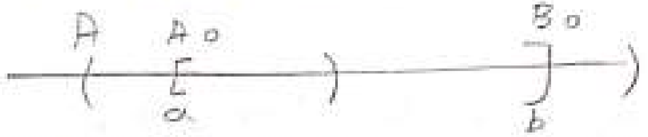
$\because$ A and B are disjoint A_0 and B_0 are also disjoint.

Thus A_0 and B_0 are open, disjoint and non-empty.

Claim 3:

$$[a,b] \neq A_0 \cup B_0$$

let c = lub in A_0



It is enough to show that $c \in A_0$ and $c \notin B_0$.

Case(i)

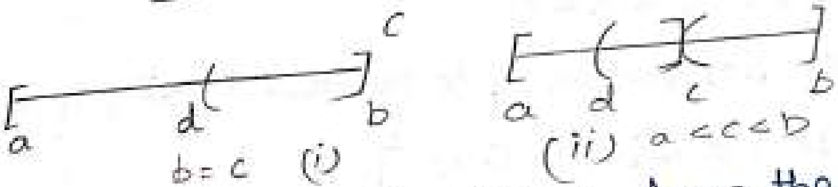
Suppose $c \in B_0$, Then c ___ a

[Because if $c = a$, Then $a \in A$, $a \in [a,b] \Rightarrow a \in A \cap [a,b]$

$\Rightarrow a \in A_0$

$\Rightarrow c \in A_0$

which is not possible because A_0 and B_0 are disjoint.]



In either case, it follows from the fact that B_0 is open in $[a,b]$ that there is some interval of the form $(d, c]$ contained in B_0.

If $c = b$ we have a contradiction at once, for 'd' is smaller upper bound and for A_0 than c. [refer fig (i)]

Case(ii)

If $c < b$ we note that $(c, b]$ does not interest A_0 [$\because$ c = l.u.b A_0]

Then $(d,b] = \underbrace{(d,c]}_{\text{Contained in } B_0} \cup \underbrace{(c,b]}_{\text{doesnot interest } A_0}$

(i.e) $(d,b]$ does not interesect A_0

Again d is a smaller upper bound on A_0.

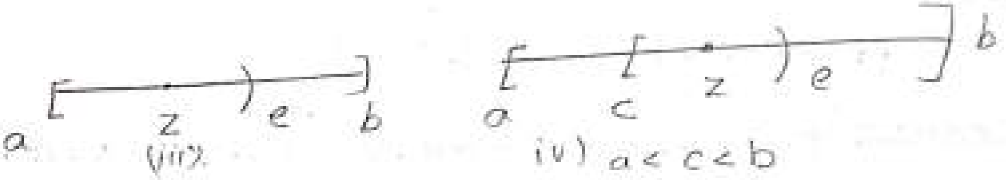
Then c, contrary to our assumption that

$$c = l.u.b\ A_0$$

Case (iii)

Suppose $c \in A_0$. Then $c \neq b$

So, either $c = a$ (or) $a < c < b$.



Because A_0 is open in $[a, b]$. There must be some interval the form $[c, e]$ contained in A_0.

Because of order property (2) of the linear continum L, We can choose a point z of L

$$\ni: c < z < e$$

Then $z \in A_0$, contrary to the fact that

$c = lub A_0$ [refer (iii) & (iv)]

Hence proved

Note:

Order relation:

A relation C on a set A is called an order relation (or) a simple order (or) a linear order if it has the follow properties.

C - relation b/w

a) Comparability:

For every x & y in A for which $x \neq y$, we have either $x \,C\, y$ ~~$x \in y$~~ (or) $y \in x$. $y \,C\, x$

b) Non - reflexivity:

For no $x \in A$ does the relation $x \,C\, x$ / $x \subset x$ hold.

c) Transitivity:

If $x \in y$, $y \,C\, z$ then, $x \,C\, z$.

Defn: An ordered set A is said to have one lub property if every non-empty subset A_0 of A that is bounded above has a lub.

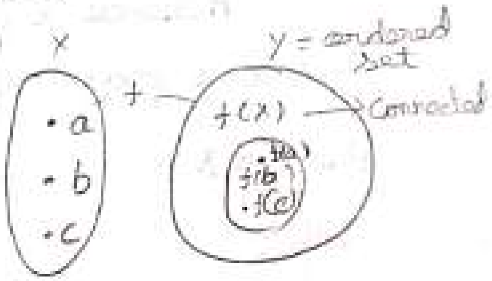
The set A is said to have the greatest lower bound property if every non-empty subset A_0 of A
(i.e) bounded below has a greatest lower bound.

Theorem 3.6 Intermediate Value theorem.

Let $f: X \to Y$ be a continuous maps of the connected space X into the ordered set Y, in the order topology. If a & b are any two points in X and if r is a point in Y (lying between $f(a)$ and $f(b)$). Then $\exists$ a point c of $X \ni: f(c) = r$.

6m

Proof:-



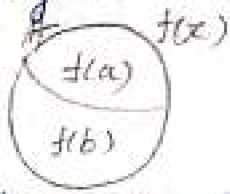
Assume the hypothesis of the theorem.

Let $A = (-\infty, r) \cap f(X)$. (open space)

$B = (r, \infty) \cap f(X)$.

where $(-\infty, r)$ and (r, ∞) are open reysis y.

Here A & B are disjoint ($\because$ the open rays $(-\infty, r)$ and $(r, -\infty)$ are disjoint).



Further A & B are open in $f(X)$.

Because they equal the intersection of an open ray of Y with $f(X)$.

Also A & B are non-empty because one contains $f(a)$ & the o____ contains $f(b)$.

Suppose if there does not exist a point r in $f(X)$.

$\ni: f(c) = r$.

Then A & B form a separation of $f(X)$. This gives a contradiction.

Because $f(X)$ is connected.

[$\because$ The image of a connected space of a under a continuous map is connected].

Hence proved.

Defn: Path

Given points x & y of the space X. Then a Path in X from x to y is a continuous map.

$f: [a,b] \to X$ of Some closed intervals of in the real line into X $\ni$: $f(a) = x$, $f(b) = y$.

A space X is said to be path connected if every pair of points of X can be joined by a Path in X.

$[a] = \{x / x \sim a$

Defn:

Given X, define an equivalance relation on X by setting $x \sim y$ if there is a connected subset of X containing both x & y.

The equivalence classes are called the Components of X.

Here symmetry and reflexivity are obvious transitivity follows by nothing noting that if A is a connected set containing x & y and B is a connected set containing y & z.

Then $A \cup B$ is a set containing x & z, which is connected because A & B have the point 'y' as common.

[by a thm the union of conneted set the have a point in common is connected].

Theorem 3.7 6m 5 17

The Components of X are connected disjoint subspace of X whose union is X, $\ni$: each connected subspace of X intersects only one of them.

Proof:-

Being components are equivalence classes, they are disjoint and their union equals X.

<u>Claim</u>: Each connected set A intersects only one of the components.

Suppose if a connected set A intersects two components C_1 & C_2 (say) at the points x_1 & x_2 respectively.

Then A is a connected set containg both x_1, & x_2, by defn $x_1 \sim x_2$.

This is possible only if $C_1 = C_2$.

$\therefore$ A intersects in only one of the compounents.

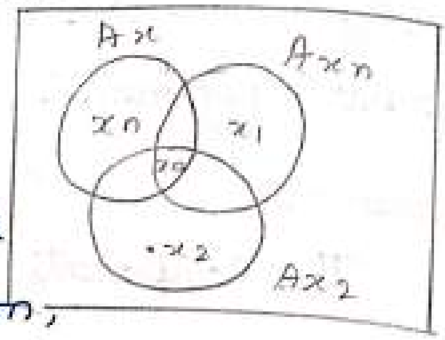
To show that the components C is connected choose a point $x_0 \in C$.

For each $x \in$ __ , we know that $x \sim x_0$, by defn,

There is a connected subset A_x of X contains both x & x_0.

of the components, we have $A_x \subset C$ —

Thus $— = \bigcup_{x \in C} A_x$



$\because$ Each A_x is connected and they have a point x_0 in common, this union C is connected.

[By a thm, the union of a collection of connected set that have a point in common is connected].

Path Components:

Defn an equivalence relation on the space X by defining $x \sim y$ if then exists a path in X from x to y.

The equivalence class are called path components of X.

Theorem: 3.8

The path components of X are path connected disjoint subspace of X where union is $X \ni:$ each non-empty paths connected subspace of X intersects to only one of them.

Proof:

Being path-components are equivalent classes they are disjoint and their union equals X.

Claim:

Each path connected set A intersects in only one of the path components.

Suppose if a path connected set A intersects two path components C_1 & C_2 (say) at the points x_1 & x_2 respectively.

Then A is path connected set containing both x_1 & x_2.

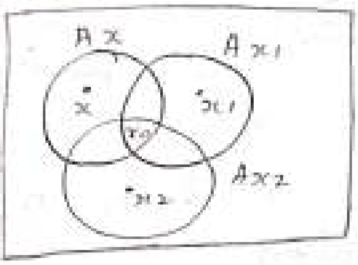
By defn $x_1 \sim x_2$. This is possible only if $C_1 = C_2$.

$\therefore$ A intersects is only one of the path components.

To show the path components C is path-connected, choose a point $x_0 \in C$. For each $x \in C$, we know that $x \sim x_0$.

By defn there is a path connected subset A_x of X contains both x & x_0.

$\because$ Each path-connected set is contained in only one of the path components we have $A_x \subset C$



Thus $C = \bigcup\limits_{x \in C} A_x$.

$\because$ Each A_x is path connected and they have a point x_0 in common, their union is path connected

[By a thm the union of the ~~two~~ collection of connected sets that have a point in common is connected].

Defn: A space X is said to be locally connected at x if for every nbh U of x there is a connected nbh V of x $\ni: V \subset U$.

If X is locally connected at each of its points.

It is said to be locally connected.

Defn:- A space X is said to be locally path-connected at x if for every nbh U of x there is a path connected nbh V of x $\ni: V \subset U$.

If X is locally path connected at each of its points. Then it is said to be locally path connected.

5m (5) (18)

Theorem 3.9:

(3) A space X is locally connected $\Leftrightarrow$ for every open set U of X, each component of U is open in X.

Proof:-

Let X be locally connected.

Let U be open in X.

Let C be a component of U in X.

<u>Claim</u>:

C is open in X.

Let $x \in C$

$\because$ U is open in X & X is locally connected, by defn.

We ~~component~~ choose a connected nbh V of x $\ni: V \subset U$.

$\because$ V is connected it must lie entirely within C.

[By a theorem, each connected set intersects only one of the components]

Thus, for each $x \in C$

we have $V_x \in C$

So $C = \bigcup_{x \in C} V_x$

$\because$ Each V_x is open and also W.K.T arbitary union of open sets is open we say that

C is open in X.

Conversely:-

Let the given hypothesis be true

<u>claim:</u> x is locally connected.

Let $x \in X$ and U be open in X containing x.

Let C be a component of U containing x.

Then C is connected.

($\because$ Each components is connected).

by hypothesis, C is open.

$\therefore$ C is a connected nbh of x contained in U.

By defn, X is locally connected at x.

$\because$ x is arbitary, we say that X is locally connected.

Theorem 3.10

A space X is locally path connected $\Leftrightarrow$ For every open set U of X, each path components of U is also open in X.

Proof:- Let X be ~~f~~ locally path connected.

Let U be open in X.

Let C be a path component of U in X.

claim:

C is open in X.

Let $x \in C$.

$\because$ U is open in X & X is locally path connected

by defn.

we can choose a path connected nbh V of x

$\ni: V \subset U$.

$\because$ V is connected it must ~~it~~ lie entirely with in C.

[By a theorem each connected set intersects only one of the components].

Thus for each $x \in C$.

we have $V_x \subset C$.

So, $C = \bigcup_{x \in U} V_x$

$\because$ Each V_x is open and also we know that arbitary union of open sets is open.

we say, that,

C is open in X.

Conversly:-

let the given hypothesis be true. let $x \in X$ and V be open in X containing x.

Let C be a path component of U containing x. Then C is path connected

($\because$ Each path components is path connected)

By hypothesis C is open.

$\therefore$ C is a path connected nbh of x contained in U.

By defn,

X is locally paths connected at x.

$\because$ x is arbitary we say that X is locally path connected.

Hence proved.

Theorem 3.11

If X is a topological Space, each path component of X lies in a componets of X. If X is locally path connected. Then the components and the path components of X are the Same.

Proof:-

Let C be a component of X.

Let x be any point in C.

Let P be a path component of X containing x.

<u>claim</u>:

$$P \subset \mathcal{C}$$

W.K.T, Each path component of X is necessiarly path connected (Thm 8) and path connected is necessiarly connected (Thrm 10)

Hence we conclude that each path-component of X is necessarily connected.

So, P is connected

By a theorem (3.8) each connected subset of X interesects only one of the components the connected set P lies in $\mathcal{C}$.

(i.e) $P \subset \mathcal{C}$.

We wish to claim that if X is locally path connected. Then $P = \mathcal{C}$

Suppose $P \neq \mathcal{C}$

Let Q be the union of all path components of X, that are different from P & intersects $\mathcal{C}$. Each of them necessarily lies in $\mathcal{C}$ So that $\mathcal{C} = P \cup Q$.

$\because$ X is locally path connected each path component of X is open & hence P (path component of X) and Q (union of path component of X) are open in X.

Further P & Q are disjoint non-empty and their union equals $\mathcal{C}$

Hence P & Q form a separation of $\mathcal{C}$.

Contradicting the fact that $\mathcal{C}$ is connected.

Thus we conclude that,

$$P = \mathcal{C}$$

Hence proved.

Unit - IV

Compact Spaces

Defn: Covering.

A collection A of subspace of a space X is said to cover X (or) a covering for X if the union of the element of A equals X.

Example: $X = \{a, b, c\}$

$$A = \{\phi, \{a\}, \{b, c\}\}$$

$$\phi \cup \{a\} \cup \{b, c\} = \{a, b, c\} = X$$

Here A is a covering for X

Def: open Covering:

The set A is called an open covering for X if its elements are open subsets of X.

Defn: Compact space.

A space X is said to be compact if each open covering ~~A~~ A of X contains a finite subcollection that also covers X.

Lemma: 4.1

Let Y be a subspace of X. They is compact iff every covering of Y sets open in X contains a finite subcollection covering of Y.

Proof:

Direct Part:

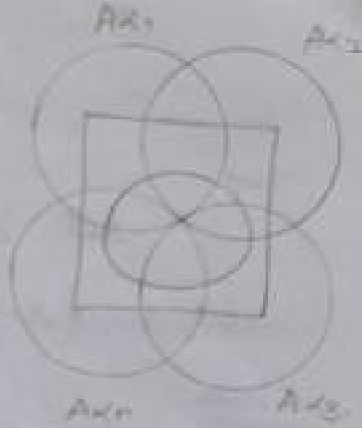
Suppose that Y is compact and

$A = \{A_\alpha / \alpha \in J\}$ be a Covering of by sets open in X

$\Rightarrow \{A_{\alpha_1}, A_{\alpha_2} \cdots\}$ covers Y.

Claim:

$\{A_{\alpha_1}, A_{\alpha_2} \ldots A_{\alpha_n}\}$ Covers Y.



The collection $\{A_\alpha \cap Y : \alpha \in I\}$ covers Y by sets open in Y.

[By the defn, of subspace topology

$\mathcal{J}_Y = \{U \cap Y / U \text{ is open in } X\}$]

$\because$ Y is compact.

$\Rightarrow$ by defn. of Compact space the subcollection

$\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, \ldots A_{\alpha_n} \cap Y\}$ covers Y.

$\therefore \{A_{\alpha_1}, A_{\alpha_2} \ldots A_{\alpha_n}\}$

which is a finite subcollection of $\mathcal{A}$ covers Y.

Converse part:

Assume that hypothesis

(i.e) Every Covering $\mathcal{A}$ of Y has a finite subcollection that also converse Y.

claim Y is compact

Let $\mathcal{A}' = \{A'_\alpha / \alpha \in J\}$ covers by sets open in Y.

For each α, choose A_α open in $X \ni: A'_\alpha = A_\alpha \cap Y$

[by defn, of subspace topology]

Now the collection,

$\mathcal{A} = \{A_\alpha / \alpha \in J\}$ be a covering of Y by sets open in X

From the assumption, $\exists$ a finite subcollection.

$\{A'_{\alpha_1}, A'_{\alpha_2} \ldots A'_{\alpha_n}\}$ covers Y.

$\Rightarrow Y$ is compact.

Theorem : 4.1

Every closed subspace of a compact space is compact.

Proof:-

Let X be a Compact space Y be a closed subspace of X.

Claim:

Y is compact. Let A be a covering of Y by sets open in X, For proving Y to be compact, it is sufficient to prove that a finite subcollection A covers Y.

(Refer Previous thm (4.1)).

Let us form an open covering B of X adjoining A to the open set X − Y.

[$\because Y \subset X$ and Y is closed

$\Rightarrow X - Y$ is open].

$B = A \cup \{X - Y\}$

Here B the open covering of X covers to space X.

$\because$ X is Compact (given) a finite subcollection of B covers X.

If the subcollection of B contains X − Y, either discard X − Y (or)

Leave the subcollection alone, this resulting collection is a finite subcollection of A that covers Y.

Theorem 4.2

Every Compact subspace of a hausdorff space is closed.

Proof.

Suppose Y be a compact space X be a Hausdorff space and Y be a subset.

Claim: Y is closed.

It is sufficient to show that $X - Y$ is open.

Let $x_0 \in X - Y$.

For each $y \in Y$ $\exists$ disjoint nbh U_y & V_y of the pts. x_0 & y respectively (using the Hausdroff Condition for the space X)

The collection $\{V_y / y \in Y\}$ is a covering for Y by sets open in X.

$\because$ Y is compact, finitely many of them cover the space Y. (by a thm) So, $\{V_{y_1}, V_{y_2}, \dots V_{y_n}\}$ cover the space Y.

Now, $V = V_{y_1} \cup V_{y_2} \cdots \cup V_{y_n}$ is disjoint from the set

$$U = U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_n}$$

Formed by taking the intersect of the corresponding nbh of x_0, because $z \in V$.

$\Rightarrow z \in V_{y_i}$ $\forall i$

$\Rightarrow z \notin U_{y_i}$ $\forall i$

$\Rightarrow z \notin U$.

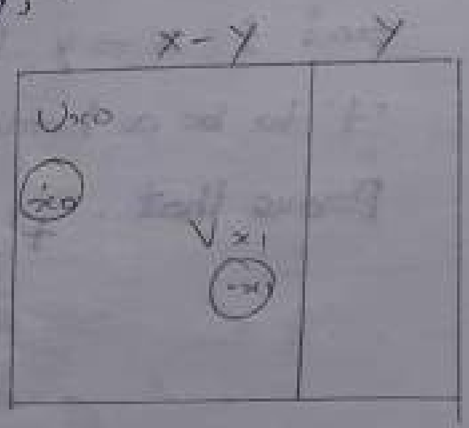
$\therefore$ U is a nbh of x_0 disjoint from Y. [$\because Y \subset V$].

$\because$ x_0 be an arbitary point in $X - Y$,

we say that $X - Y$ is open.

Hence Y is closed

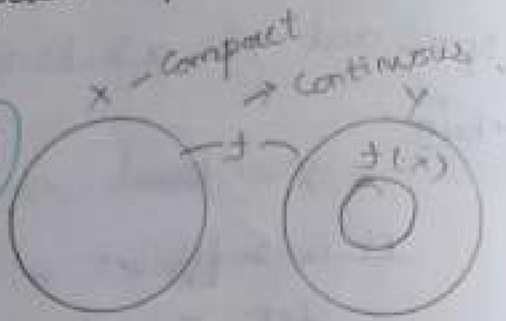
$X - Y = U_{x_0} \vee U_{x_1} \vee U_{x_2} \cdots$



Thm: 4.3

The image of a compact space under a continuous map is compact.

2 times (16) (6m)



Proof:-

Claim:

$f(X)$ is Compact.

Suppose $\{A_\alpha / \alpha \in J\}$ is a covering for $f(X)$ by sets open in Y.

So, $\{f^{-1}(A_\alpha) / \alpha \in J\}$ cover X.

$\because f$ is continuous by defn for each A_α open in Y, $f^{-1}(A_\alpha)$ is open in X.

$\because X$ is Compact by defn finitely many of them cover the space.

(ie) $f^{-1}(A_{\alpha_1}), f^{-1}(A_{\alpha_2}) \ldots f^{-1}(A_{\alpha_n})$ cover.

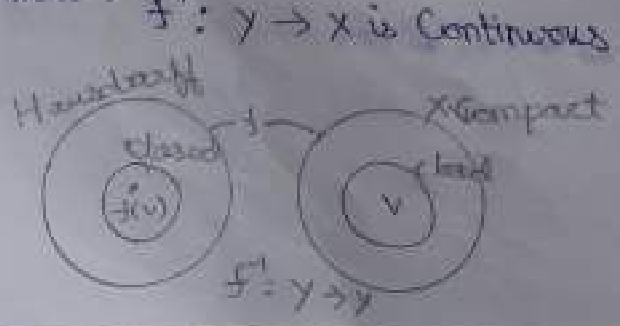
Hence $\{A_{\alpha_1}, A_{\alpha_2} \ldots A_{\alpha_n}\}$ covers $f(X)$ Thus, $f(X)$ is Compact.

Thm 4.4

Let $f: X \to Y$ be a bijective continuous ~~form~~ funs. If X is compact and Y is Hausdorff then f is homeomorphism.

Proof:-

Let $f: X \to Y$ be a bijective ~~form~~ function and $f: X \to Y$ be continuous (given) For proving 'f' to be a homeomorphism it is sufficient to prove that. $f^{-1}: Y \to X$ is Continuous.



Suppose V be closed in X, Since every closed subset of a Compact space is Compact.

we Say that V is Compact.

$\therefore f$ is Continuous from, $X \to Y$ and by a thrm, the Continuous image of a Compact space is compact, we say that $f(V)$ is also compact.

$\therefore$ Every Compact, subset of a Hausdorff space is closed, we say that $f(V)$ is closed in Y.

Hence $f^{-1}: Y \to X$ in Continuous

Hence proved.

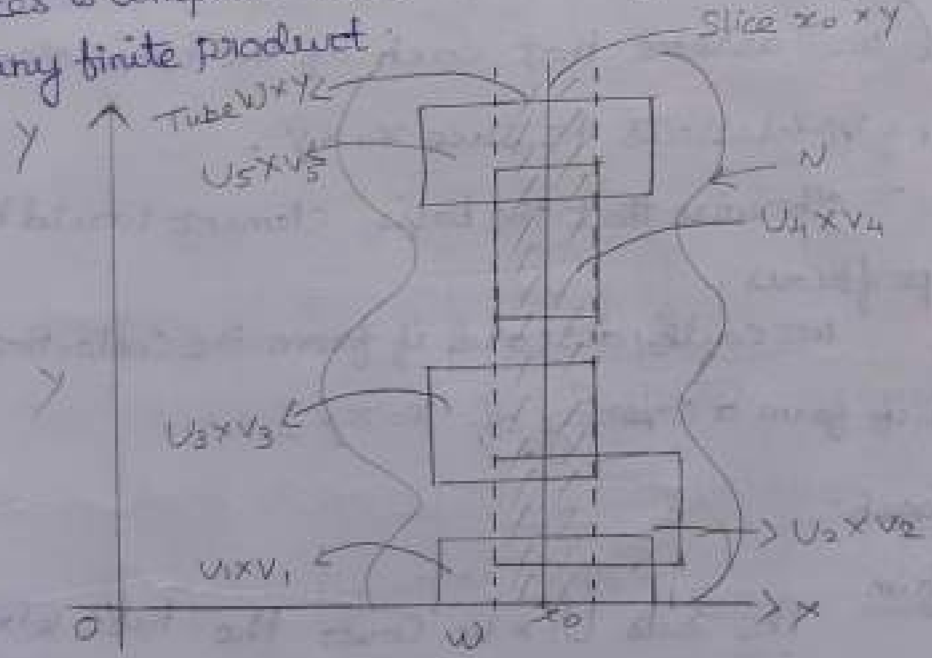
15m

Thrm: 4.5

The Product of finitely many compact spaces is Compact.

Proof:-

we Shall Prove the product of the compact spaces is compact these the thm, follows by induction for any finite product.



Step 1: Suppose that X and Y are topological spaces with Y compact. Let $x_0 \in X$ and N is an open set in $X \times Y$. Containing the slice $x_0 \times Y$ of $X \times Y$.

Claim:

There is a nbh w of x_0 in such a way that N Contains the entire set $W \times Y$.

Stage a:-

Claim $x_0 \times Y$ is Covered by $U_i \times V_i$. Let us Cover the Slice $x_0 \times Y$ by means of basis elements $U_i \times V_i$ [$U_i \times V_i$ is a basis elements of the product topology in $X \times Y \ni U_i$ is open in X and V_i is open in Y] each is lying in N.

The Slice $x_0 \times Y$ is Compact being homomorphism to Y.

Hence, finitely many of the collection taken from $U_i \times V_i$ (Say).

$U_1 \times V_1, U_2 \times V_2, \ldots U_n \times V_n$ covers the slice $x_0 \times Y$.

(we assume that each of the basis elements $U_i \times V_i$ intersects the slice $x_0 \times Y$.

$\because$ otherwise that the basis element would be superfluous.

we could discard it from the Collection and still it form a covering of $x_0 \times Y$).

Stage b:

Claim: The sets $U_i \times V_i$ Cover the tube $W \times Y$.

Define, $W = U_1 \cap U_2 \ldots \cap U_n$.

W is open in X. Also $x_0 \in W$ [$x_0 \in U_i, \forall i$]

Let $X \times Y \in W \times Y$.

Consider the pt $x_0 \times y$ on slice $x_0 \times y$. It is having the same, y co-ordinate as the pts $x \times y$

Now, $x_0 \times y \in U_i \times V_i$ for some i

$\Rightarrow$ $y \in V_i$ for that i.

$\because x \in W$ $\quad$ ($\because x \times y \in W \times y$).

we have, $x \in U_j \; \forall j$

$\because (W = U_1 \cap U_2 \dots \cap U_n)$.

Hence $x \in U_i$ for that i

$\therefore$ $x \times y \in U_i \times V_i$ for that i. Hence the collection of the product sets $U_i \times V_i$ cover the tube $W \times y$.

Stage 'c'

$\because$ all the sets $U_i \times V_i$ cover the tube $W \times y$ and each $U_i \times V_i$ is lying in N, we get $W \times y$ is covered by N.

Step 2:

Let X & Y be two compact spaces. Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$.

$\because$ The slice $x_0 \times Y$ is compact finitely many elements $A_1, A_2, \dots A_m$ of $\mathcal{A}$ covers $x_0 \times Y$. This union.

$N = A_1 \cup A_2 \dots A_m$ is an open set in $X \times Y$, containing the slice $x_0 \times Y$.

By using step. 1 the open set N contain a tube $W \times Y$ about the slice $x_0 \times Y$ where W is open on X.

Then $W \times Y$ is covered by finitely many elements $A_1, A_2, \ldots A_m$ of $\mathcal{A}$.

$[\because N = A_1 \cup A_2 \cup \ldots \cup A_m]$.

Thus for each $x \in X$, we can choose a nbh W_x of $x \ni$: tube $W_x \times Y$ is covered by finitely many elements of $\mathcal{A}$.

Let the collection of all nbd's W_x is an open covering of X.

$\therefore$ From Compactness of X, $\exists$ finite subcollection $\{W_1, W_2, \ldots W_k\}$ cover the space X.

Now,

$$X \times Y = (W_1 \times Y) \cup (W_2 \times Y) \ldots \cup (W_k \times Y).$$

$\because$ Each tube $W_i \times Y$ is covered by finitely many elements of $\mathcal{A}$.

So is the case for their union.

Hence a finite number of elements in $\mathcal{A}$ also covers $X \times Y$.

Thus $X \times Y$ is Compact.

Hence Proved.

Note: (Tube lemma).

Consider the product space $X \times Y$, where Y is compact.

If N is an open set on $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$.

Then N contains some tube $W \times Y$ of $x_0 \times Y$ where W is a nbh of x_0 in X.

Proof:- As in step 1 in the previous theorem.

Defn:-

A collection $\mathcal{C}$ of subsets of X is said to satisfy the finite intersection condition if for every finite subcollection $\{C_1, C_2, \dots C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap C_2 \cap C_3 \dots \cap C_n$ is non-empty [F.I.P] (2)

Thm : 26.

Let X be a topological space. Then X is compact iff for every collection $\mathcal{C}$ of closed sets in X, having the finite intersection Property the intersection $\bigcap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{C}$ is non empty.

Proof:-

Given a collection $\mathcal{A}$ of subsets of X,

let $\mathcal{C} = \{X - A / A \in \mathcal{A}\}$ be the collection of their complements. Then the following statements hold

i) $\mathcal{A}$ is a collection of open sets iff $\mathcal{C}$ is a collection if closed sets.

ii) The collection $\mathcal{A}$ covers X iff their intersection $\bigcap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{C}$ is empty

iii) The finite subcollection $\{A_1, A_2 \dots A_n\}$ of $\mathcal{A}$ covers X iff the intersection of the corresponding elements $C_i = X - A_i$ of $\mathcal{C}$ is empty

[from De Morgan's law;

$$X - \left\{\bigcup_{\alpha \in J} A_\alpha\right\} = \bigcap_{\alpha \in J} (X - A_\alpha) = \phi$$

Also $$X - \left\{\bigcup_{i=1}^{n} A_i\right\} = \bigcap_{i=1}^{n} (X - A_i) = \phi$$

$\because \bigcup_{\alpha \in J} A_\alpha$ covers the space X, we have

$$X - \left\{\bigcup_{\alpha \in J} A_\alpha\right\} = \phi \;]$$

The proof of the theorem is based on

i) Contrapositive of the theorem.

ii) Complement of the sets.

The Statement that X is Compact is equivalent of saying that "Given any collection λ of open sets of X, if λ covers X, Then Some finite subcollection of λ ~~covers~~ Covers X".

The Contrapositive of the above statement is given in the following way "If any collection of open sets of X is given and if no finite sub-collection λ of λ covers X, T

Then λ does not Cover X".

Applying Conditions (i), (ii) and (iii), we get the following statement.

"X is compact iff given any collection $\mathcal{C}$ of closed sets in X, if every finite sets in X, if every finite intersection of elements of $\mathcal{C}$ is non empty,"

Then the intersection of all elements of $\mathcal{C}$ is also non-empty".

Hence the theorem.

Thm: 4.7 [Theorem based on finite intersection Property] [Extension]

A special Case of the above theorem occurs when we have a nested sequence $C_1 \supset C_2 \supset C_3 \ldots \supset C_n$ of closed sets in a Compact Space X.

If each of the sets C_n is non-empty, then the collection
$\mathcal{C} = \{C_n\}_{n \in \mathbb{Z}_+}$ satisfies the finite intersection condition (H)

Then the intersection $\bigcap_{n \in \mathbb{Z}_+} C_n$ is non-empty.

Compact Sets in the real line

Theorem: 4.8.

Let X be a simply ordered set having the least upper bound property, In the order topology, each closed interval in X is compact.

Proof:-

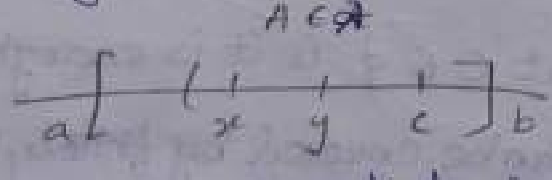
Given $a<b$.

Let $\mathcal{A}$ be a covering of $[a,b]$ by sets open in $[a,b]$ in the subspace topology.

<u>Claim:</u> A finite subcollection of $\mathcal{A}$ covers $[a,b]$.

<u>Step 1:</u> 1st we prove that following:

If $x \in [a,b]$ different from b, Then $\exists$ a point $y>x$ in $[a,b]$ $\ni$ the closed interval $[x,y]$ is covered by a atmost two elements of λ.



If x has an immediate successor in X, let y be this immediate successor

Then $[x,y]$ consists of the two point x and y, so that it can be covered by atmost two elements of λ.

Suppose that x has no immediate successor in X.

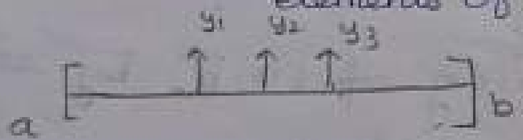
choose an element $A \in \lambda$ containing x. Because $x \neq b$ (given) and A is open, A contains an interval of the form $[x, c)$ for some $c \in [a,b]$.

choose a point y in (x, c). Then $[x, y]$ is covered by a single element A of $\mathcal{A}$.

<u>Step 2</u>:

Let $\mathcal{G}$ be the set of a points $y > a$ of $[a,b]$ $\ni$ $[a,y]$ can be covered by finitely many elements of λ.

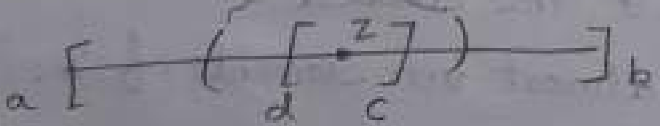
i.e $\mathcal{G} = \{ y > a \mid [a,y]$ is covered by finitely many elements of $\lambda \}$



Apply step① to the case $x = a$, we see that $\exists$ atleast one such y.

(i.e) $\mathcal{G}$ is non-empty.

<u>Step 3</u> Let c be the least upper bound of the set $\mathcal{G}$, Then we have $a < c \leq b \ni A \in \mathcal{A}$



we show that $c \in \mathcal{G}$ i.e it is enough to show that $[a,c]$ can be covered by finitely many elements of $\mathcal{A}$.

Choose an element A of $\mathcal{A}$ containing c.

$\therefore$ A is open, by defn, of open set A contains an interval of the form $(d, c]$ containing the point c for some $d \in [a,b]$

If $c \notin \mathcal{G}$, There must be

<u>Suppose $c < b$:</u>

Apply step (1) to the case $x = c$. Then there exists a point $y > c$ in $[a,b]$ $\ni$: $[c,y]$ is covered by atmost two elements of $\mathcal{A}$.

$\because c \in \mathcal{G}$ using step(3), $[a,c]$ can be covered by finitely many elements of $\mathcal{A}$.

$\therefore [a,y] = [a,c] \cup [c,y]$ can be covered by finitely many element of $\mathcal{A}$. By defn, of we have $y \in \mathcal{G}$, contradicting the factor that $c =$ lub. a point z of $\mathcal{G}$ lying in the interval (d,c) because otherwise "d" would be a smaller upper bound on $\mathcal{G}$ than c.

$\because z \in \mathcal{G}$, the interval $[a,z]$ can be covered by finitely many (say) n elements of $\mathcal{A}$. Further $[z,c]$ lies in single element A of $\mathcal{A}$.

Thus $[a,c] = [a,z] \cup [z,c]$ can be covered by $(n+1)$ elements of $\mathcal{A}$. By defn, of $\mathcal{G}$, we have $c \in \mathcal{G}$ contrary to our assumption

<u>Step 4:</u>

Finally, we have to show that $c = b$ so that the theorem is proved.

This contradiction arises only because of assuming that $c < b$

Hence we've $c = b$ i.e.) $[a,b]$ is covered by finitely many elements of $\mathcal{A}$

i.e. $[a,b]$ is compact.

Thm : 4.9 A ~~subset~~ space A of $\mathbb{R}^n$ is compact iff it is closed and bounded in the Euclidean metric 'd' on the square metric 'ρ'.

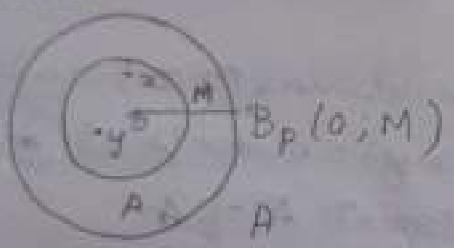
<u>Proof</u>: It is enough to consider only the square metric 'ρ' because

$\rho(x,y) \le d(x,y) \le \sqrt{n}\,\rho(x,y)$ imply that A is bounded under 'd' iff it is bounded under 'ρ'.

Direct path
Suppose A is compact w.r.t, $\mathbb{R}^n$ is a Hausdorff space, (∵ Simple ordered set is on Hausdorff space in the order topology).

∵ A is a subset of $\mathbb{R}^n$ (ie) $A \subseteq \mathbb{R}^n$ we have A is closed (by a theorem Every compact subset of a Hausdorff space is closed).

<u>claim</u>. A is bounded.



$A \subseteq B_\rho(0;M)$

Consider the collection of all open sets $B_\rho(0;M)$

∋: each $M \in \mathbb{Z}^+$

ie $\{B_\rho(0;M) / M \in \mathbb{Z}^+\}$ where union is all of $\mathbb{R}^n$

[∵ $B_\rho(0;M) = \{y / \rho(0,y) < M\}$]

∵ A is compact, some finite collection ρ-balls cover A.

∴ We have $A \subseteq B_\rho(0;M)$ for some M.

∴ for any two pts $x, y \in A$, we have

$$\rho(x,y) \le \rho(x,0) + \rho(0,y)$$
$$\le M + M$$
$$\le 2M$$

Hence A is bounded.

Conversely suppose A is closed and bounded under the metric ρ.

<u>Claim</u>: A is compact

Converse Part

A is bounded, $\rho(x,y) \le N$ for every pair of points $x,y \in A$. Let us choose a point $x_0 \in A$ and $\rho(x_0, 0) = b$ from the triangle law of inequality, we have,

ie $\rho(x,0) \le \rho(x,x_0) + \rho(x_0,0)$

$\le N + b \quad \forall x \in A$

$[\because \rho(x,y) \le N \quad \forall x,y \in A$

In particular put $y = x_0 \in A$

$\therefore \rho(x,x_0) \le N \quad]$.

Put $P = N + b$.

So for every $x \in A$, we have $\rho(x,0) \le P$.

$\therefore A$ is a subset of $[-P, P]^n$

(ie) $A \subset [-P, P]^n$

Here the closed interval $[-P, P]$ is compact.

Hence $[-P,P]^n$ is also compact [$\because$ Product of finitely many compact spaces is also compact].

$\therefore A$ is closed and being a subset of a compact space $[-P,P]^n$, by a theorem, we've A is compact.

6m 5 '18 15m

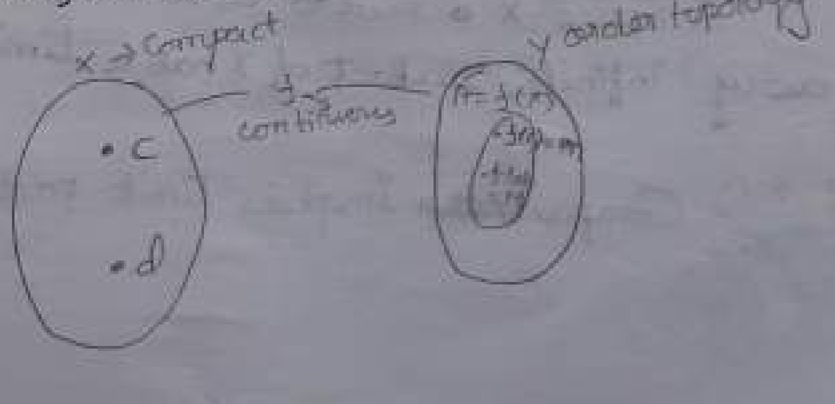
2-times 3-ti

Thm: 4.10 Extreme value theorem

Maximum and minimum value theorem

Let $f: X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, Then $\exists$ points c, d in $X \ni: f(c) \le f(x) \le f(d) \ \forall x \in X$.

Proof:-



It is given that X is Compact and $f: X \to Y$ be continuous. T

$\because$ The image set $A = f(X)$ is also Compact.

[by a theorem, " The image of a compact space under a continuous map is compact "].

Next we've to S.T A Contains a largest element 'M' and a Smallest element 'm'. Then,

$\because$ m and M belong to A, we must have

$$m = f(c)$$
$$M = f(d) \text{ for some } c, d \in X.$$

Suppose A has no largest element. Then the collection $\{(-\infty, a) / a \in A\}$ forms an open covering of A.

$\because$ A is Compact, finitely many of then cover A. (say) $(-\infty, a_1), (-\infty, a_2) \dots (-\infty, a_n)$ cover A.

If a_i is the largest element in $a_1, a_2, \dots a_n$, then a_i belongs to none of the above sets, Contradicting the fact that they cover A.

A Should possess the largest element say $f(d) = M$.

A Similar argument S.T A has a smallest element say $f(c) = m$.

Hence the theorem.

Defn:-

A space X is said to be limit point Compact if every infinite subset of X has a limit point.

Thm 4.11 Compactness implies limit point Compact.

6m (i6) S

Proof: Let X be a Compact space. Given a subset A of X. we wish to prove if A is infinite, then A has a limit point.

We prove it's contrapositive. If A has no limit point, Then A must be finite.

Suppose A has no limit. Then A contains all its limit points. Hence A is closed.

[∵ A subset A of a topological space X is called closed iff it contains all its limit points].

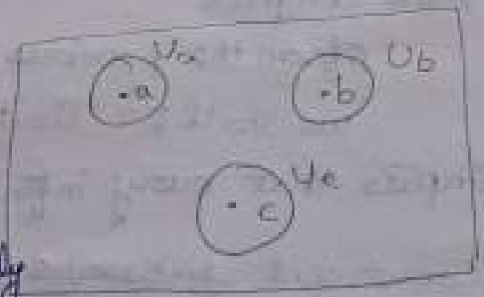
∵ Every closed subset of a compact space is compact, we say that A is compact.

For each $a \in A$, we can choose a nbd U_a of 'a' ∋: U_a does not intersect $A - \{a\}$,

∴ 'a' is not a limit point of A,

[∵ x is a limit pt of A, Then every nbd U of x intersect $A - \{x\}$].

ie $U_a \cap [A - \{a\}] = \phi$.



The set A is covered by the open sets U_a.

using A is compact, finitely many of the open sets U_a cover A

Thus A is covered by (say) some 'n' finite open sets.

∵ U_a Contains only one point of A, we've the set A contains 'n' points. Thus A is finite.

Hence X is a limit point compact

Defn:- The diameter of a bounded subset A of a metric space (X, d) is the number lub $\{d(a_1, a_2) / a_1, a_2 \in A\}$

Def:- If every sequence in space Y has a convergent subsequence, we say that Y is sequentially compact.

4.12 & 4.1

Thm: 4.12 (The Lebesgue number lemma).

Let λ be an open covering of the metric space (X, d). If X is Compact, Then there is a $\delta > 0$ $\ni$: for each subset of X having diameter less than δ, Then $\exists$ an element of λ containing it.

["The number δ is called a lebesgue number for the covering λ.]

Proof

Step 1:

Because X is Compact, It necessarily limit point compact.

(from the previous thm).

we shall P.T limit point compactness of X inturn implies that every infinite sequence (x_n) of X has a convergent subseque.

i.e) There is an increasing sequence $n_1 < n_2 < \dots n_i < \dots$ of positive Integers $\ni$: The sequence $x_{n_1}, x_{n_2}, \dots, x_{n_i} \dots$ converges

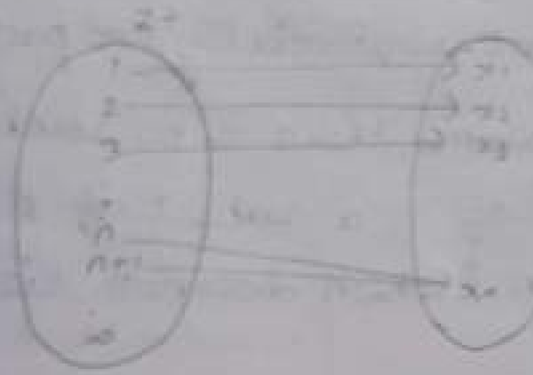
[we have to S.T limit point compact $\Rightarrow$ Sequentially compact].

Given the sequence (x_n). Let us consider the set $A = \{x_n / n \in Z^+\}$.

First, suppose that the set A is finite

In this Case, we assert that there is a pt, x $\ni$: $x = x_n$ for infinitely many values of n

[To Prove this assertion, let $f: Z_+ \to A$ be defined by $f(n) = x_n$. Then Since Z_+ is the union of the finite collection of sets $f^{-1}(x)$, as x ranges over A, atleast one of the sets $f^{-1}(x)$ must be infinite].



Here $f^{-1}(x_n) = \{n, n+1, n+2, \dots\}$ infinite set

Then the sequence (x_n) has a subsequence that is Constant which converges automatically.

Second, suppose that A is infinite. Then A has a limit pt 'x', [$\because$ X is limit point compact].

we define a subsequence of (x_n) converging to 'x' as follows:

First choose n_1 so that $x_{n_1} \in B(x, 1)$. Then suppose that the positive integer n_{i-1} is given because the ball $B(x, \frac{1}{i})$ intersects A in infinitely many points, we Can choose an index $n_i > n_{i-1}$ $\ni$: $x_{n_i} \in B(x, \frac{1}{i})$

Then the subsequence $x_{n_1}, x_{n_2} \dots$ coverging to x.

Step 1 If X is compact Then X is limit pt, Compact.

If X is limit pt, compact, Then X is sequentially compact, when there A is finite (or) infinite

Step 2: Now we S.T if X is sequentially compact Then every open covering $\mathcal{A}$ of X has a lebesgue number δ.

we shall prove the contrapositive:

Suppose there is no $\delta > 0$ $\ni$: Every set of diameter less than δ lies in atleast one element of $\mathcal{A}$.

Then X is not sequentially compact.

So let us assume there is no such δ. This means that for each $\delta > 0$, $\exists$ a subset of X having diameter less than δ which does not lie inside any element of $\mathcal{A}$.

In particular, for each $n \in \mathbb{Z}_+$ we can choose a set C_n having diameter less than $\frac{1}{n}$ which is not contained in any element of $\mathcal{A}$.

choose, for each n, a point 'x_n' of C_n we assert that the sequence (x_n) has no convergent subsequeness.

Suppose that sequence (x_n) had a convergent subsequence (x_{n_i}), convergeing to x

Now x belongs to some element A of $\mathcal{A}$ and because A is open, There is an $\epsilon > 0$ $\ni$: $B(x, \epsilon) \subset A$

Choose i larger of enough that
$d(x_{n_i}, x) < \epsilon/2$ and $\frac{1}{n_i} < \frac{\epsilon}{2}$

Now C_{n_i} lies in a $\frac{1}{n_i}$ nbd of x_{n_i}. It follows that $C_{n_i} \subset B(x, \epsilon)$.

Then $C_{n_i} \subset A$, Contradicting the choice of the sets.

Thm : 4.13 : Uniform Continuity theorem : 4, 14 (Y) (15 M)

Statement : Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the ~~Comp~~ metric space (Y, d_Y). Then f is uniformly continuous. (6 M) (9) (a)

(ie) for a given $\epsilon > 0$ $\exists$ a $\delta > 0$ $\ni$: for any two pts. x_1, x_2 in X and $d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$ (M)

Proof :-

Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$.

Let $\mathcal{A}$ be an open covering of X by the inverse image of these balls under f

[$\because f$ is continuous].

Choose δ to be a lebesgue number for the covering $\mathcal{A}$. Then if x_1, x_2 are any two pts in X $\ni$: $d_X(x_1, x_2) < \delta$, the two pt set (x_1, x_2) has diameter less than δ, So that it's image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$

Then from triangles inequality, we've

$$d_Y\{f(x_1), f(x_2)\} < \epsilon$$

Hence the theorem

$f(x_1) \in B(y, \epsilon/2)$ and $f(x_2) \in B(y, \epsilon/2)$ [say]

(ie) $d(f(x_1), y) < \epsilon/2$

$d(f(x_2), y) < \epsilon/2$

$$= d_Y(f(x_1), f(x_2))$$

$$< d_Y(f(x_1), y) + d_Y(f(x_2), y)$$

$$< \epsilon/2 + \epsilon/2$$

$$< \epsilon$$

$$\mathcal{A} = \{f^{-1}(B(y), \epsilon/2)), f^{-1}(B(y_1, \epsilon/2)), f^{-1}(B(y_2, \epsilon/2)) \ldots\}$$

Thm 4.14. 6m S (1)

Let x be a metrizable space. Then the following are equivalent

i) x is compact.

ii) x is limit point compact.

iii) x is sequentially compact.

Proof:- (i) => (ii) (proved)

(ii) => (iii) (proved).

Suppose X is sequentially compact.

Claim. x is compact.

W.k.t if X is sequentially compact, Then every open covering A of x has a lebesgue number δ.

[Second part].

Step 1: First we S.T $\forall \epsilon>0$, $\exists$ a finite covering of x by ϵ-balls.

Let us take it's contrapositive. If for some $\epsilon>0$, The space X cannot be covered by finitely many ϵ-balls.

Then X is not sequentially compact.

So suppose that X cannot be covered by finitely many ϵ-balls.

Construct a sequence of pts x_n as follows.

First choosen x_1 to be any point in X. noting that the ball $B(x_1,\epsilon)$ is not all of x

[Otherwise X is covered by this single ϵ-ball (viz) -> $B(x,\epsilon)$]

Choose x_2 to be a point of X which is not in $B(x_1,\epsilon)$.

In general given $x_1, x_2 \ldots x_n$

choosen the pt x_{n+1} not in the

union.

$B(x_1, \epsilon) \cup B(x_2, \epsilon) \ldots \cup B(x_n, \epsilon)$,

using the fact that X Cannot be covered by

these ϵ - balls.

note that by construction $d(x_{n+1}, x_i) \geq \epsilon$ for $i = 1, 2 \ldots n$

Therefore, the sequence (x_n) has no convergent subsequence

In fact, any ball of radius $\epsilon/2$ can contain 'x_n' for

atmost one value of n.

As a Consequence X is not sequentially Compact.

Step 2

Now we prove X is compact. let $\mathcal{A}$ be an open covering of X.

$\because$ X is sequentially compact, using previous theorem,

every open covering $\mathcal{A}$ of X has a lebesgue number δ.

using Step 1, Choose a finite covering of X by balls of

radius $\delta/3$.

Each of these balls has diameter atmost $\frac{2\delta}{3}$,

Some can choose for each of these balls and element

of $\mathcal{A}$ containing it.

we Thus obtain a finite subcollection of $\mathcal{A}$

that Covers X.

Tietze extension thm. continuous

(for proof) $f : X \to [a, b]$

$f : X \to [a, b]$

$f(x) = a \ \forall x \in A$

$f(x) = b \ \forall x \in B$

$g : X \to [-\frac{1}{3}r, \frac{1}{3}r]$

$g(x) = -\frac{1}{3}r \ \forall x \in B$

$g(x) = \frac{1}{3}r \ \forall x \in A$

$$g(x) = \sum_{n=1}^{\infty} g_n(x) \quad \forall\, x \in X$$

$$|g(x)| = \left| \sum_{n=1}^{\infty} g_n(x) \right|$$

$$= \sum_{n=1}^{n} |g_n(x)|$$

$$\leq \frac{1}{3} + \frac{1}{3}\left(\frac{2}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right)^2 + \cdots$$

$$\leq \frac{1}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right]$$

[Geometric progression $S_\infty = \frac{a}{1-r}$, if $r<1$]

A Geometric series from comparison

Real analysis $1 + r + r^2 + \cdots$ converges if $r<1$

and diverges if r ...

Def: Isolated Point:-

If x is a space, a point x of x is said to be an isolated point of X if the one point set $\{x\}$ is open in x.

Thm: 4.15

Let X be a non-empty compact Haustaff space. If X has no isolated points, then X is an uncountable.

Proof:-

Step 1: First we S.T given any non-empty open set U of x and any point x of x, if a non-empty open set $V \subset U$ $\ni$: $x \notin \bar{V}$

Let us choose a point $y \in U$ different from x. This is possible because of two condition

i) If $x \in U$ because x is not isolated point $x, y \in U$

ii) If $x \notin U$, then since U is non-empty, $y \in U$

$\therefore$ X is Hausdorff, choose disjoint open sets W_1 and W_2 about x & y respectively, then the set $V = W_1 \cap U$ is the desired open set.

If it is Contained in V_y it is non empty Since it Contains y and it's closure does not Contain x.

Step 2

To S.T $f: Z_+ \to X$ is not Surjective

$\Rightarrow X$ is uncountable.

Let $x_n = f(n)$.

Apply step I to the non-empty open set $U = X$.

To choose a non-empty open set $V_1 \subset X \ni$ $\overline{V_1}$ does not contains x_1.

In general V_{n-1} is an open set choosen V_n to be non empty open set $\ni$. $V_n \subset V_{n-1}$ & $\overline{V_n}$ does not Contain x_n.

Consider a nested sequence

$\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \ldots$ of non-empty closed set of X.

$\because X$ is Compacts, there is a pt $x \in \cap \overline{V_n}$

By a theorem, $x \neq x_n$ for any n

(i.e) $x_n \notin X$,

$\Rightarrow f(x_n) \notin X$

i.e f is not surjective.

$\Rightarrow X$ is uncountable

Local Compactness.

A space X is said to be locally Compact at 'x'. if there is some compact subspace C of X that Contains a neighbourhood of x.

If X is locally Compact at each of its Point then X is said to be locally Compact.

Definition.

If Y is a Hausdroff compact space and X is a proper subspace of Y. whose closure equals Y Then Y is said to be compact_fication of X. If $Y-X$ equals, a single point then Y is called the one point compaktification of X.

Example for locally compact.

The Real line $\mathbb{R}$ is locally compact The point x lies in some interval (a,b) which interm is contained in the compact subspace $[a,b]$ The subspace $\mathbb{Q}$ of Rational Numbers is not locally compact.

5 ③ 6m

Theorem:

Let X be a Hausdroff space then X is locally compact iff Given x in X and given neighbourhood U of x, there is a neighbourhood V of x $\ni:$ $\bar{V}$ is compact and $\bar{V} \subset U$.

Proof:-

Given $x \in X$ and $\exists$ a neighbourhood U of x, $\exists$ a neighbourhood V of x.

And $\bar{V} \subset V$ and neighbourhood V of x

$\therefore$ X is locally compact.

Conversely,

Suppose X is locally compact.

Let $x \in X$ and U be a neighbourhood of x. Now taken the one point compartification Y of X.

Let $C = Y - U$ Then C is closed in Y,

$\Rightarrow$ C is compact subspace of Y.

Then by lemma,

If Y is compact subspace of Hausdroff space X and x_0 is not in Y. Then $\exists$ disjoint open sets U and V of X containing x_0 and Y respectively.

choosen disjoint open sets U and W containing x and C respectively.

Then the ~~cho~~ closure $\bar{V}$ of V in Y is compact Also $\bar{V}$ is disjoint from C.

$\therefore \bar{V} \subset U$.

Hence proved.

Corollary:-

Let X be locally compact Hausdroff let A be a subspace of X. If A is closed in X or open in X Then A is locally compact.

Proof:-

Suppose that A is closed in X

Given $x \in A$, let C be a compact subspace of X containing the neighbourhood U of x in X.

Then $U \cap A$ is closed in C and thus compact And it contains the neighbourhood $U \cap A$ of x in A.

Now, suppose that A is open in X.

Given $x \in A$,

we apply above theorem, to choose a neighbourhood V of x in X.

$\ni$: $\bar{V}$ is compact and $\bar{V} \subset A$.

Then $C = \bar{V}$ is a compact subspace of A containing the neighbourhood V of x in A

Hence the proof.

Unit - V

Countability axioms.

Countable Basis:

A space X is said to have a countable basis at a point 'x' if there is a countable collection $\mathcal{B}$ of neighbourhood of x $\ni$: Each neighbourhood of x contains atleast one of the element of $\mathcal{B}$

A space that has a countable basis at each of its points is said to satisfy the first countability axiom of to be first - countable.

2^{nd} Countability Axiom:

If a space X has a countable basis for its topology then X is said to satisfy the 2^{nd} countability axiom or to be 2^{nd} countable.

Ex: Any matrix space X is a First - Countable.

Proof:-

Let $x \in X$.

Let $\mathcal{B} = \{B(x, 1/n);\ n = 1, 2 \dots\}$

Now let $x \in U$, which is open then there exist a $\epsilon > 0$ $\ni: B(x, \epsilon) \subset U$.

Now choose $n \ni: 1/n < \epsilon$

$\Rightarrow B(x, 1/n) < B(x, \epsilon) \subset U$

Thus X is First - Countable.

Theorem:

A subspace of a 1^{st} Countable space is first countable and a countable product of first - Countable space is first countable.

A subspace of a 2nd countable space is 2^{nd} Countable and a Countable product of 2^{nd} Countable space is 2^{nd} Countable.

Suppose X is first Countable.

Let A be a subspace of X and let $x \in A \Rightarrow x \in X$.

$\because$ X is first - countable $\exists$ a Countable basis $\mathcal{B}$ for x in X.

Then $\mathcal{B}' = \{B \cap A \,/\, B \in \mathcal{B}\}$ is a Countable basis x in A.

Now $B \in \mathcal{B}$.

$\Rightarrow B$ is open in X and $x \in B$.

$\Rightarrow B \cap A$ is also open in A and $x \in B \cap A$

$\Rightarrow B \cap A$ is a neighbourhood of x in A.

$\therefore$ $\mathcal{B}$ is Countable.

$\Rightarrow$ $\mathcal{B}'$ is also countable

$\therefore$ $\mathcal{B}'$ is a countable collection of neighbourhood of x in A.

Finally if U is any neighbourhood of x in A Then $\exists$ an open set V in X $\ni: U = V \cap A$

$\therefore$ $\mathcal{B}$ is a Basis at x, $\exists B \in \mathcal{B} \ni: x \in B \subset V$

$\Rightarrow x \in B \cap A \subset V \cap A = U$

Hence $\mathcal{B}'$ is a countable basis at x in A

$\therefore$ A is First - countable.

Let $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$, for each α fixed a Countable basis at x_α in X_α.

$\mathcal{B} = \{\prod B_\alpha \,/\, B_\alpha \in \mathcal{B}_\alpha\}$ for a finite number of α's and $B_\alpha = X_\alpha$ $\forall$ other α's Then $\mathcal{B}$ is Countable basis at x.

Hence Product of X_α, ie, $\prod X_\alpha$ is also first Countable.

Now we prove for 2^{nd} Countable,

Let $\mathcal{B}$ be a Countable basis for a topology Then $\{B \cap A \,/\, B \in \mathcal{B}\}$ is a countable basis for the subspace A

of X if $\mathcal{B}_i$ is a Countable basis for the space X_i

Then the collection of all product πU_i

where $U_i \in \mathcal{B}_i$ for finitely many values of and $U_i = X_i$ for all other values of i is a countable basis basis for πX_i

$\therefore \pi X_i$ is a 2^{nd} Countable.

$\because x_n$ is arbitary

$\Rightarrow x \subset \bar{D}$ and also $\bar{D} \subset X$.

$\Rightarrow \bar{D} = X$.

Dense set:

A subset A of a space X is said to be dense in X if $\bar{A} = X$.

Theorem:

Suppose that X has a countable basis than

a) Every open covering of X contains a countable subcollection covering X.

b) There exists a countable subset of X that is dense in X.

Proof:

Let $\{B_n\}$ be a countable basis for X.

To prove:

Part a) Let $\mathcal{A}$ be an open covering of X for each +ve integer n choose an element A_n of $\mathcal{A}$ containing B_n.

$\because$ it is indexed with a subset J of +ve integers, the collection $\mathcal{A}'$ of the set A_n is countable.

Now we prove that $\mathcal{A}'$ covers X.

Let $x \in X$, we can choose an element A of cover $\mathcal{A}$ containing at a point x.

$\because A$ is open, $\exists$ a basis element $\ni$: $x \in B_n \subset A$

$\because B_n \subset A \in \mathcal{A}$

Then the index n belongs to the set J So A_n is defined

$\because B_n \subset A_n \Rightarrow x \in A_n$

Thus A' is a cover of x and

$\therefore A'$ is a countable subcollection of A that cover x.

To Prove : Part b) :

Choose a point x_n from each non-empty basis element B_n

Let D be the set containing the point x_n.

To prove that D is dense in X.

(i.e) we S.T every non-empty open set in X intersects with D.

Given any point $x_n \in X$ and let U be any neighbourhood of x_n in X.

$\because B$ is a basis $\exists$ an element $B_n \in \mathcal{B} \ni x_n \in B_n \subset U$.

$\Rightarrow x_n \in D \cap U$

$\Rightarrow D \cap U \neq \phi$

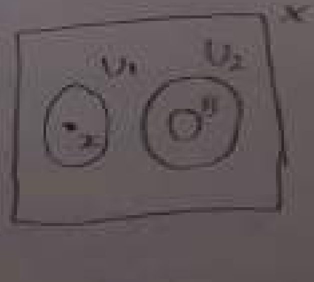
Lindelof Space.

A space for which every open covering contains a countable subcovering Lindelof space.

Separable :

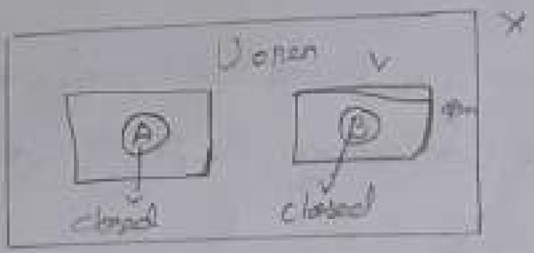
A space having a countable dense subset is called separable. The separation Axioms.

Regular: Suppose that one point sets are closed in X. Then X is said to be regular if for each pair consisting of a point x of X and a closed set B disjoint from x, $\exists$ disjoint open sets containing x and B respectively.

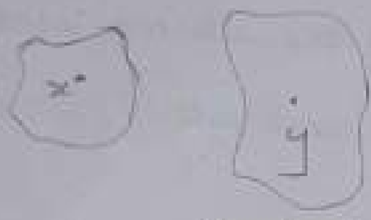


Normal:

The space X is said to be normal if for each pair A and B of disjoint closed sets of X $\exists$ disjoint open sets containing A and B respectively.



Hausdorff



Lemma: Let X be a topological space and let one point sets in X be closed.

a) X is regular iff given a point x of X and a neighbourhood U of x $\exists$ a neighbourhood V of x $\ni: \bar{V} \subset U$.

b) X is normal iff given a closed set A and an open set U containing A, there is open set V containing A $\ni: \bar{V} \subset U$.

Proof:-

To prove a)

Let X be regular.

Given $x \in X$ $\exists$ a neighbourhood U of x. Let $B = X - U$

$\Rightarrow$ B is a closed set in X.

Now by hypothesis, $\exists$ disjoint open sets U and W containing x and B respectively.

i.e.) $x \in V$ and $B \in W$ or $B \subset W$

$\Rightarrow V \cap W = \phi$

$\Rightarrow V \subset X - W$

$\Rightarrow V \subset X - B$

$\Rightarrow V \subset U$ $\quad \therefore x \in V \subset \bar{V} \subset U$

i.e.) $\bar{V} \subset U$.

Conversely,

Let $x \in X$ and choose a closed set B not containing x.

Suppose $U = X - B$ by assumption, $\exists$ neighbour-hood V of $x \ni \bar{V} \subset U$.

$\because \bar{V}$ is closed

$\Rightarrow X - \bar{V}$ is open

$\therefore$ The open sets V and $X - \bar{V}$ are disjoint containing x and V

$\therefore X$ is regular.

To prove (b):

Let X be normal given A is closed set and $\exists$ an open set U containing x in A.

we choose $E = X - U$

$\Rightarrow E$ is a closed set

$\because X$ is normal, $\exists$ an open set V, W of $X \ni A \subset V$, $E \subset W$ and also $V \cap W = \phi$

$\Rightarrow V$ contained is $X - W$

$\Rightarrow V \subset X - W$

$\Rightarrow V \subset X - E \subset U$

$\therefore V \subset U$

$A \in V \subset \bar{V} \subset U$

$\therefore \bar{V} \subset V$.

Conversely, let us choose a closed set A of X and a closed set B not containing A. we choose $U = X - B$

$\Rightarrow U$ is an open set containing A

By assumption, $\exists$ an open set V containing $A \ni \bar{V} \subset U$

$\because \bar{V}$ is closed

$\Rightarrow X - \bar{V}$ is open

(ie) The open set V and $X - \bar{V}$ are disjoint containing the closed sets A and B

$\therefore X$ is normal.

Theorem. 5m

a) A subspace of a Hausdroff space Hausdroff. a product of a Hausdroff space is Hausdroff.

b) A subspace of a completely regular space is regular a Product of a regular space is regular.

Proof:-

To prove a):

Let X be a Hausdroff space

Let Y be a subspace of X.

Let $x, y \in X$

$\because$ X is Hausdroff

$\Rightarrow \exists$ two disjoint neighbourhood U and V of x and y in X.

$\Rightarrow x \in U$ and $y \in V$

$\Rightarrow$ Then $U \cup Y$ and $V \cap Y$ are disjoint neighbourhood of x and y in Y.

$\therefore$ Y is Hausdroff.

Now we prove that the product of Hausdroff space of Hausdroff.

Let $\{X_\alpha\}$ be a family of Hausdroff space Let $x = (x_\alpha)$ and $y = (y_\alpha)$ be distrinct of the product space ΠX_α

$\because x \neq y$, $\exists$ Some index $\beta \ni: x_\beta \neq y_\beta$ choose disjoint open sets U and V in X_β containing x_β and y_β Then the sets Πx_β and Πy_β are also disjoint open sets in ΠX_α containing x and y.

$\therefore$ The sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open set in ΠX_α containing x and y

$\therefore \{X_\alpha\}$ is Hausdroff Space.

To prove (b).

Let X be a regular, space and Y be a subspace of α.

∴ By using definition of regular

"one point sets are closed in Y".

Let $x \in Y$ and let B be a closed subset of Y disjoint from x

∴ we choose $\overline{B} \cap Y = B$. where $\overline{B}$ is the closure of B.

∴ $x \notin \overline{B}$

By using regularity of X,

choose disjoint open set U and V containing x and $\overline{B}$.

Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B respectively.

∴ Y is regular.

Let $\{X_\alpha\}$ be a family of regular spaces and let $X = \Pi X_\alpha$ [∵ by a)]

X is Hausdorff so that one point sets are closed in X

Now we prove that $X = \Pi X_\alpha$ is regular let $x = (x_\alpha)$ be a point of X and let U be a neighbourhood of $x \in X$.

choose a basis element ΠU_α about x containing in U for each α,

choose a neighbourhood V_α of x_α in X_α $\ni: \overline{V_\alpha} \subset U_\alpha$

If $U_\alpha = X_\alpha$ Then choose $V_\alpha = X_\alpha$.

∴ $V = \Pi V_\alpha$ is a neighbourhood of x in X

∵ $\overline{V} = \Pi \overline{V_\alpha}$ is follows that $\overline{V} \subset \Pi U_\alpha \subset U$

∴ X is regular

15m 5 18

Normal space.

Theorem: Every regular space with a Countable basis is normal.

Proof:-

Let X be a regular space with a countable basis $\mathcal{B}$.

Let A and B be disjoint closed subsets each point x in X

$\Rightarrow$ Each point of x of A has a neighbourhood U not intersecting B.

By using definition of Regularity, choose a neighbourhood V of x where closure lies in U

(ie) $\overline{V} \subset U$.

Finally choose an element of basis $\mathcal{B}$ containing x and contained in V.

By choosing such a basis element for each x in A, we construct a countable covering of A by open sets whose closures do not intersects B.

Now let us denote it by $\{U_n\}$

lly choose a countable collection $\{V_n\}$ of open sets covering B $\ni$: each $\overline{V_n}$ is disjoint from A.

Now $U = \cup U_n$ and $V = \cup V_n$ are the open sets containing A and B respectively.

But they need not be disjoint.

Let us construct two open sets that are disjoint for given n, we define

$U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_i}$ and $V_n' = V_n - \bigcup_{i=1}^{n} \overline{U_i}$

Each set U_n' is open being the difference of an open set U_n and closed set $\bigcup_{i=1}^{n} \overline{V_i}$

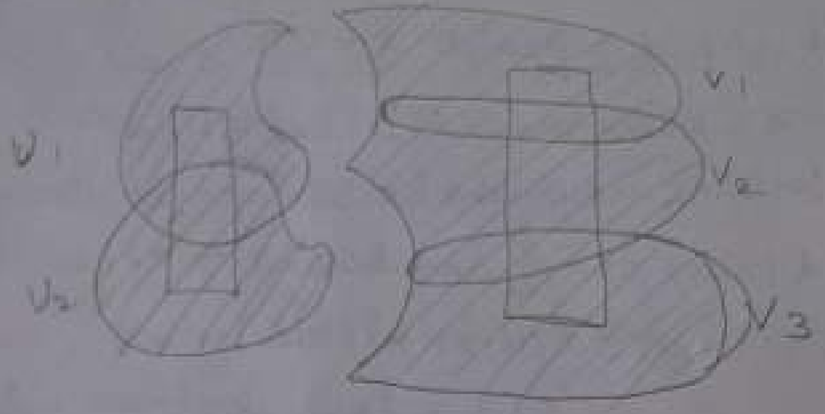
lly Each set V_n' is open.

Now x is $A \in U_n$, for some n and $x \in$ to none of the sets $\overline{V_i}$

$\therefore$ The collection sets $\{U_n'\}$ covers A

lly The collection sets $\{V_n'\}$ covers B.

Finally we have to prove that the open sets $U' = \bigcup_{n \in Z^+} U_n'$ and $V' = \bigcup_{n \in Z^+} V_n'$ are disjoint.



For if $x \in U' \cap V'$ Then $x \in U_j' \cap V_k'$ for some j and k.

Suppose that $j \leq k$ from the definition of U_j' $x \in U_j$ and since $j \leq k$ and Also from the definition of V_k'.

$$x \notin \overline{U}_j$$

$$\text{ie) } V_k' = V_k - \bigcup \overline{U}_j$$

which is a $\Rightarrow\Leftarrow$

A Similar contradiction arises if $j \geq k$

Hence $U' \cap V' = \phi$

5 a) (5m)

Theorem

5. Every metrizable space is Normal.

M **Proof**: Let X be a metrizable space with metric 'd'.

Let A and B be disjoint closed subset of X.

For each $a \in A$, choose ϵ_a so that the ball $B(a, \epsilon_a)$ does not intersect B.

Illy For each $b \in B$ choose ϵ_b so that the ball $B(b, \epsilon_b)$ does not intersect A.

Define $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$ and

$$V = \bigcup_{b \in B} B(b, \epsilon_b/2)$$

Then U and V are open sets containing A and B respectively.

Now we prove that $U \cap V = \phi$

For, if $U \cap V \neq \phi$ Then let $z \in U \cap V$

ie) $z \in B(a, \epsilon a/2) \cap B(b, \epsilon b/2)$

for Some $a \in A$ and $b \in B$.

$\Rightarrow d(z,a) < \epsilon a/2$ and $d(z,b) < \epsilon b/2$

Now $(a,b) \leq d(a,z) + d(z,b)$

$< \epsilon a/2 + \epsilon b/2$

$< \epsilon a/2 + \epsilon b/2$

If $\epsilon_a \leq \epsilon_b$ Then $d(a,b) < \epsilon_b$

The ball $B(b, \epsilon_b)$ Contains The point 'a'

If $\epsilon_b \leq \epsilon_a$ Then $d(a,b) < \epsilon_a$

$\therefore$ The ball $B(a, \epsilon a)$ contains The point 'b'

$\because$ A and B are disjoint.

This is a Contradiction.

$\therefore U \cap V \neq \phi$

$\therefore$ X is normal.

5 6m 18 17 16 3-times

Theorem:

Every Compact Hausdroff space is Normal.

Proof:-

Let X be a Compact Hausdroff space.

Let A and B be two disjoint closed subsets of X.

$\therefore$ X is compact and A and B are closed.

$\Rightarrow$ A and B are also compact.

By Lemma,

"If Y is a compact space of the Hausdroff space X and x_0 is not in Y, Then $\exists$ disjoint open sets U and V of containing x_0 and Y respectively".

Choose for each point $a \in A$ disjoint open sets U_a and V_a Containing A and B respectively.

The collection $\{U_a\}$ Covers A.

$\because$ A is compact, 'A' may be covered.

By finitely many sets $V_{a_1}, V_{a_2}, \ldots V_{a_m}$. Then

$U = U_{a_1} \cup U_{a_2} \cup \ldots \cup U_{a_m}$ and

$V = V_{a_1} \cup V_{a_2} \cup \ldots \cup V_{a_m}$ are disjoint open sets containing A and B respectively.

$\therefore X$ is normal.

Theorem: Every well ordered set X is normal in the order topology.

Proof:- Let X be a well ordered set.

we prove that every interval of the form $[x, y]$ is a open in X.

If y is the largest element of X then $[x, y]$ is just a basis element about y.

If y is not the largest element of X then $[x, y]$ equals the open set (x, y')

where y' is the immediate successor of y.

Let A and B be disjoint closed set in Y.

Let us assume that neither A nor B contains the smallest element a_0 of X.

For each $a \in A$, $\exists$ a basis element about 'a' disjoint from B.

It contains some interval of the form $[x, a]$ choose for each $a \in A$, such an interval $[x_a, a]$ disjoint from B.

Illy for each $b \in B$, choose an interval $[y_b, b]$ disjoint from A.

Then the sets $U = \bigcup_{a \in A} (x_a, a]$ and

$V = \bigcup_{b \in B} (y_b, b]$ are open sets containing A and B respectively.

Now we prove that $U \cap V = \phi$.

For If $U \cap V \neq \phi$ Then $z \in U \cap V$.

$\Rightarrow z \in [x_a, a] \cap [y_b, b]$ For some $a \in A$, $b \in B$

Assume that $a < b$,

If $a \leq y_b$ Then the two intervals are disjoint.

If $a > y_b$ then $a \in [y_b, b]$

Contrary to the fact that $[y_b, b]$ is disjoint from A.

A Similar Contradiction occurs if $b < a$

$\therefore U \cap V = \phi$.

Finally assume that A and B are disjoint closed set in X

let A contains a smallest element a_0 of X.

The set $\{a_0\}$ is both open and closed in X.

By above Result, $\exists$ disjoint open-sets U and V containing the closed set $A \{a_0\}$ and B respectively.

Then $U \cup \{a_0\}$ and V are disjoint open sets Containing A and B respectively.

$\therefore$ X is Normal.

The Urysohn Lemma

Statement: Let X be a normal space. Let A and B be disjoint closed subset of X. Let $[a, b]$ be a closed interval in the real line. Then $\exists$ a continuous map $f: X \rightarrow [a, b] \ni$:

$f(x) = a$ for every x in A and

$f(x) = b$ for every x in B.

Proof:- without loss of generality, instead of $[a, b]$ we consider the interval $[0, 1]$.

Step 1. Let P be the set of all rational numbers in the interval $[0, 1]$.

For each $p \in P$, we shall define an open set U_p of X $\ni$: whenever $p<q$, $\overline{U_p} \in U_q$.

The sets U_p will be simple ordered.

$\because$ P is countable, we can use induction to define the set U_p.

Arrange the elements of P is an infinite sequence

Let us suppose that the number 1 and 0 be the first two elements of the sequence.

Now define the sets U_p as follows.

First define $U_1 = X - B$.

$\Rightarrow$ A is contained in U_1

By Normaly of X, choose an open set U_0 $\ni$: $A \subset U_0$ and $\overline{U_0} \subset U_1$.

In general let P_n denote the set consisting of the 1st rational numbers is the sequence.

Suppose that U_p is defined for all rational $p \in P_n$ satisfying the condition $p<q \Rightarrow \overline{U_p} \subset U_q$. — (*)

Let r denote the next rational number is the sequence.

Now we define U_r.

consider the set $P_{n+1} = P_n \cup \{r\}$ P_{n+1} is a finite subset of the interval $[0,1]$. Also it has a simple ordered set on the Real line.

"In a finite simply ordered set, Every element (other than the smallest and the largest) has an immediate Predecessor and successor".

In the simply ordered set P_{n+1} the number 0 is the smallest element and 1 is the largest element and r is neither 0 nor 1.

$\therefore$ r has an immediate processor p in P_{n+1} and an immediate successor q in P_{n+1}. The set U_p and U_q are already defined and $\overline{U_p} \subset U_q$ by the induction hypothesis.

Using Normality of X, we can find an open set U_r of $X \ni: \overline{U_p} \subset U_r$ and $\overline{U_r} \subset U_q$.

we claim that ⊛ holds for every pair of elements of P_{n+1}

In both the elements lie in P_n ⊛ by induction hypothesis.

If one of them is r and other is a point S of P_n.

ie) $r, S \in P_n$

If $S \leq p$ Then $\overline{U_S} \subset \overline{U_p} \subset U_r$

If $S \geq q$ Then $\overline{U_r} \subset \overline{U_q} \subset U_S$.

Thus every pair of elements of P_{n+1} ⊛ holds.

By induction, we have, U_p defined $\forall\, p \in P \ni:$ ⊛ holds

Step ②:

In step 1, we define U_p $\forall$ rational numbers p in the interval $[0,1]$.

we extended this definition to all rational numbers in $\mathbb{R}$.

By defining, $U_p = \phi$ if $p < 0$

$U_p = X$ if $p > 1$.

For every pair of Rational number p and q, $p < q$,

$\Rightarrow \overline{U_p} \subset U_q$.

Step ③:

Let $x \in X$ and define $Q(x)$ to be the set of those rational number $p \ni:$ The corresponding open sets U_p contain x.

i.e) $Q(x) = \{P / x \in U_p\}$

Then for $p<0$, No number less than 0.

Also $P>1$, every x is in U_p.

$\therefore$ This set contains every number greater than 1.

$\therefore$ $Q(x)$ is bounded below and its g(t) is a point of the interval $[0,1]$

Define $f: X \to [0,1]$ by

$$f(x) = \inf Q(x)$$
$$= \inf \{P / x \in U_p\}.$$

<u>Step 4</u>: W.K.T f is the desired function.

If $x \in A$ Then $x \in U_p$ for every $p \geq 0$

$\Rightarrow Q(x)$ equals the set of all non-negative rationals and $f(x) = \inf Q(x) = 0$

Il'ly if $x \in B$ Then $x \in U_p$ for no $p \leq 1$.

$\Rightarrow Q(x)$ Consist of all rational numbers greater than 1 and $f(x) = 1$.

Now we prove that f is continuous.

For this we prove the following elementary facts,

i) $x \in \overline{U}_r \Rightarrow f(x) \leq r$

ii) $x \notin \overline{U}_r \Rightarrow f(x) \geq r$

i) If $x \in \overline{U}_r$ Then $x \in U_r$ for every $S > r$.

$\therefore$ $Q(x)$ Contains all rational numbers greater than r.

$\therefore$ By definition we have

$$f(x) = \inf Q(x) \leq r.$$

ii) If $x \notin U_r$ Then $x \notin U_s$ for every $S < r$.

$\therefore$ $Q(x)$ Contains no rational number less than r

So that by definition we have.

$$f(x) = \inf Q(x) \geq r.$$

Now we Prove continuity of f

Let $x_0 \in X$ and let (c,d) be an open interval in $\mathbb{R}$ Containing the point $f(x_0)$

To prove that, $\exists$ a neighbourhood U of x_0 $\ni: f(U) \subset (c,d)$

choose rational numbers p and q $\ni: c < p < f(x_0) < q < d$.

we Prove that the open set $U = U_q - \overline{U_p}$ is the desired neighbourhood of x_0.

First we note that $x_0 \in U$. For the fact that $f(x_0) < q$.

=> by condition ii) That $x_0 \in U_p$ while $f(x_0) > p$ implies by (i) That $x_0 \notin \overline{U_p}$.

Hence $x_0 \in U_q - \overline{U_p}$ U

Next we Show that $f(u) < (c,d)$.

Let $x \in U$. Then $x \in U_q \subset \overline{U_q}$ So that $f(x) \leq q$

And $x \notin \overline{U_p}$ So that $x \notin U_p$, $f(x) \geq p$.

Thus $f(x) \in [p,q] \subset [c,d]$.

$\therefore$ f is continuous at $x_0 \in X$.

$\because$ x_0 is arbitrary.

f is continuous on X.

Hence proved.